Prize-collecting Survivable Network Design in Node-weighted Graphs

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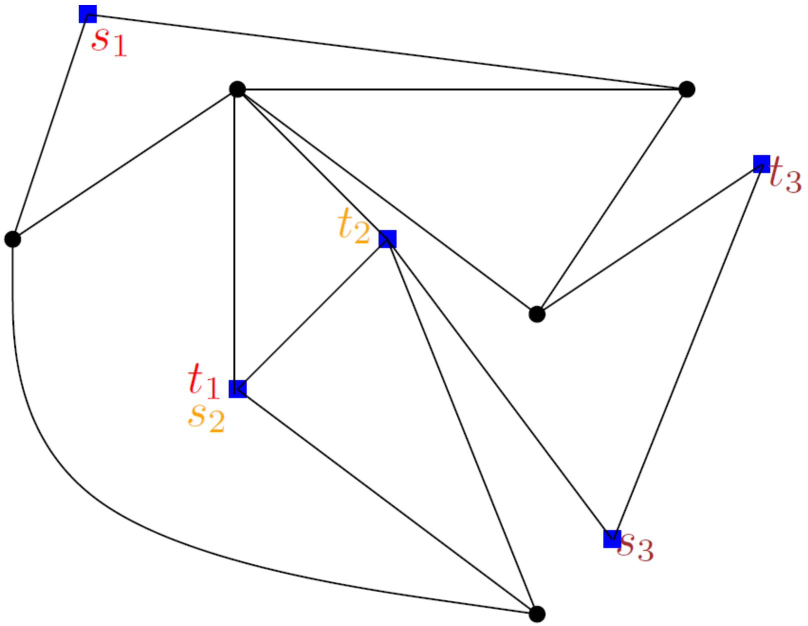
Survivable Network Design (SNDP)



Collection of l pairs: $(s_1, t_1), \ldots, (s_l, t_l)$

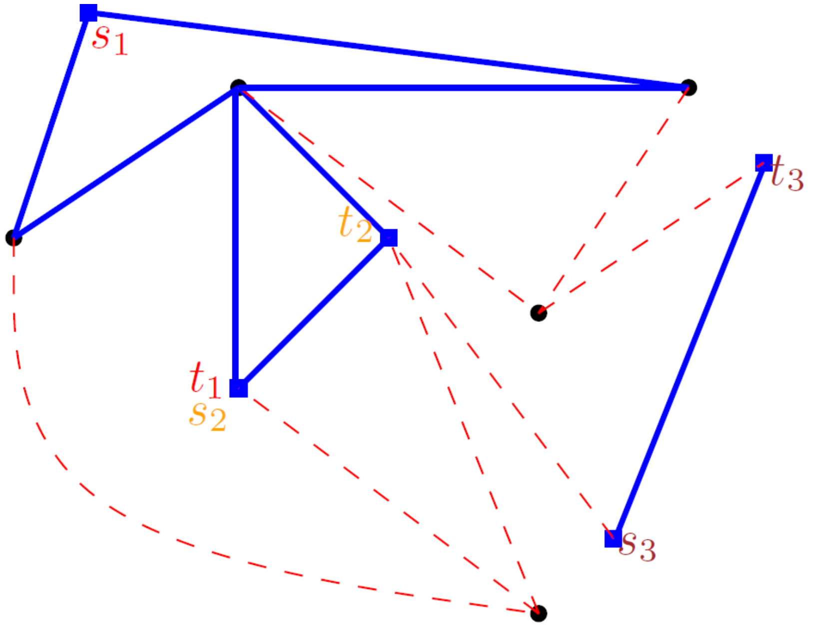
$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \leq r} r(s_i, t_i)$

Goal: Min the sum of the weight of subgraph H containing $r(s_i, t_i)$ disjoint paths between s_i and t_i.

$$r(s_1, t_1) = 2$$
$$r(s_2, t_2) = 2$$
$$r(s_3, t_3) = 1$$

Survivable Network Design (SNDP)



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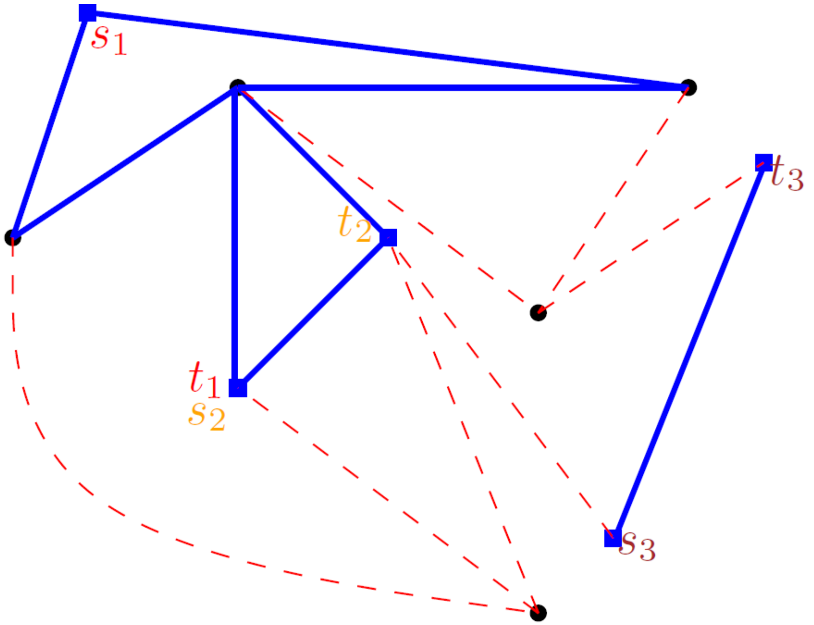
Well-known special cases: Steiner tree/forest (k=1)

Survivable Network Design (SNDP)



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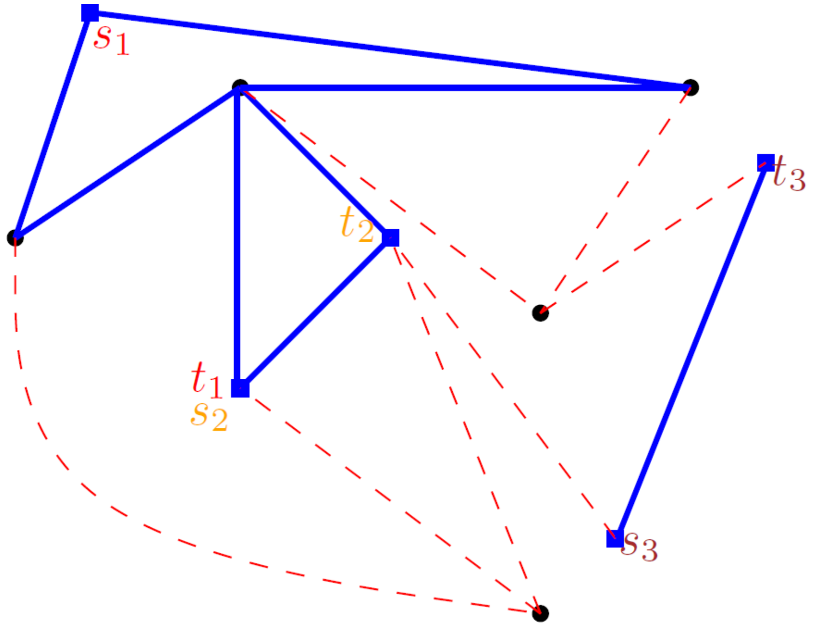
	Edge dt.	Element dt.	Vertex dt.
Edge wt.	2-approx Jain '98	2-approx Fleischer et al. '01	$O(k^3 \log n)$ Chuzhoy-Khanna '09
Node wt.	$\Omega(\log n)$-hard for Steiner tree Klein and Ravi '95		

Survivable Network Design (SNDP)



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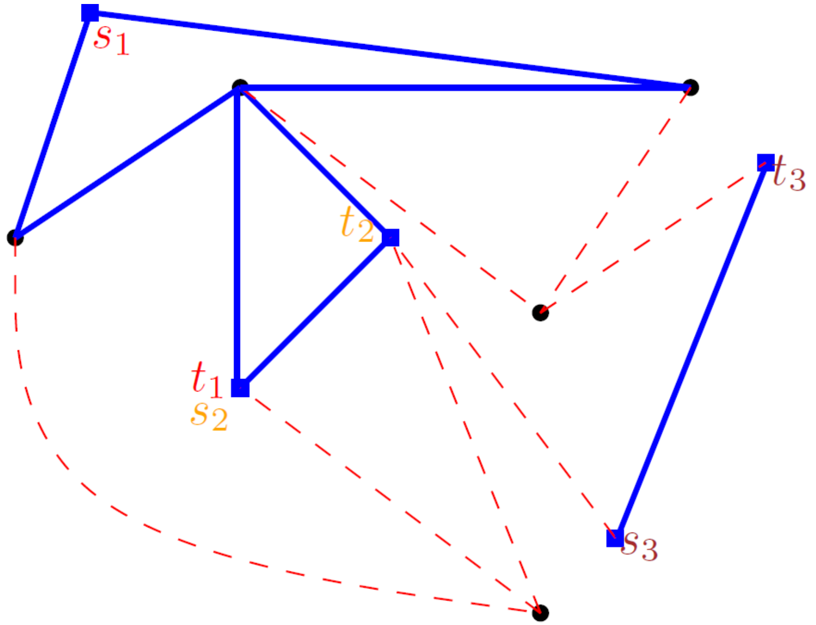
Node wt.	$O(k \log n)$ Nutov '08	$O(k \log n)$ Nutov '09	$O(k^4 \log^2 n)$ Nutov '09

Prize-collecting Survivable Network Design (PC-SNDP)



Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \leq r} r(s_i, t_i)$

$\pi(s_i, t_i)$: penalty for not satisfying the connectivity of pair (s_i, t_i)

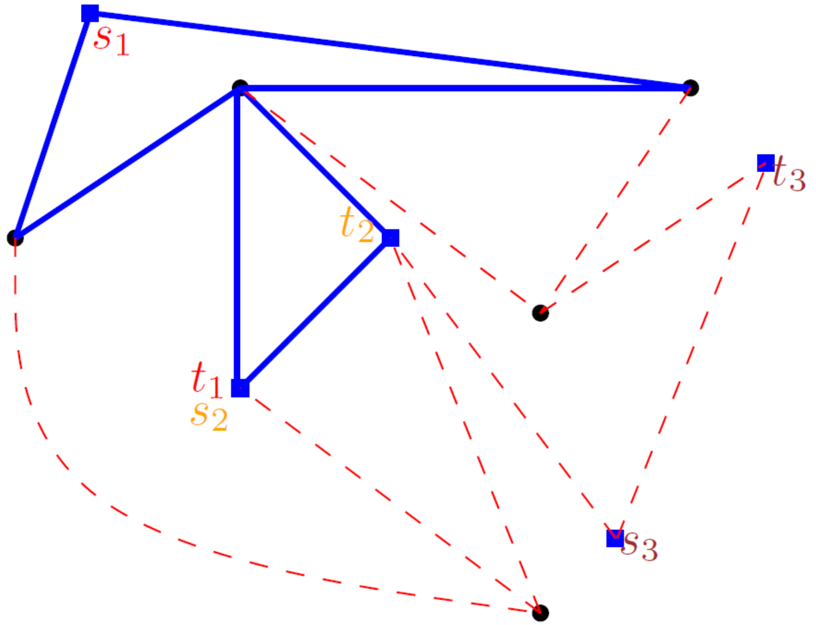
Goal: Min the sum of the weight of subgraph H + *the sum of penalties for requirements not satisfied by* H.

$r(s_1, t_1) = 2$ $\quad \pi(s_1, t_1) = 10$

$r(s_2, t_2) = 2$ $\quad \pi(s_2, t_2) = 20$

$r(s_3, t_3) = 1$ $\quad \pi(s_3, t_3) = 2$

Prize-collecting Survivable Network Design (PC-SNDP)



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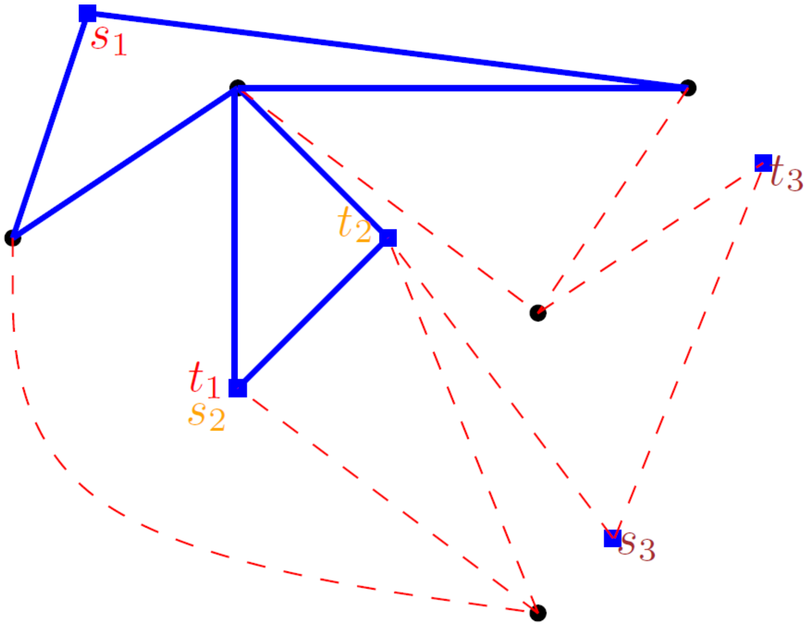
All-or-nothing penalty version

Prize-collecting Survivable Network Design (PC-SNDP)



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	Edge dt.	Element dt.	Vertex dt.
Edge wt.	2.54-approx Hajiaghayi et al. '10	2.54-approx Hajiaghayi et al. '10	$O(k^3 \log n)$ Hajiaghayi et al. '10
Node wt.	No result	No result	No result

Our Result

First approximation for node weighted PC-SNDP

	Edge dt.	Element dt.	Vertex dt.
Node wt.	$O(k^2 \log n)$	$O(k^2 \log n)$	$O(k^5 \log^2 n)$
	$O(k \log n)$*	$O(k \log n)$*	$O(k^4 \log^2 n)$*

In planar graphs [Chekuri et al. ‘12]:

	Edge dt.	Element dt.	Vertex dt.
Node wt.	$O(k^2)$	$O(k^2)$	$O(k^5 \log n)$
	$O(k)$*	$O(k)$*	$O(k^4 \log n)$*

*Running time is polynomial in n^k

Multiroute-flow based LP relaxation for (PC-)SNDP

- No LP-relaxation for node-weighted SNDP was known before.

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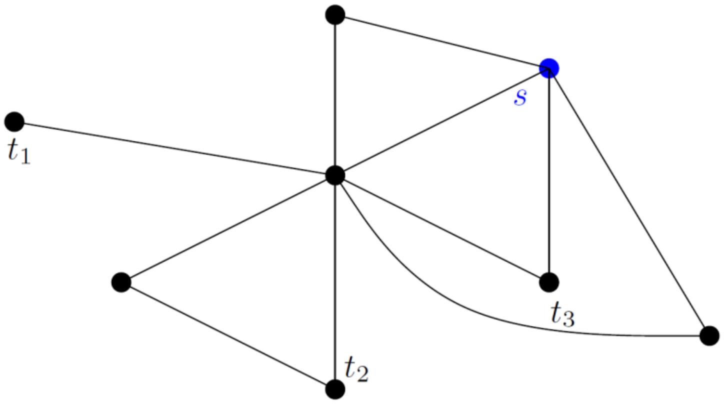
PC-Steiner Tree

Edge weighted, edge connectivity



R: Set of Steiner nodes

Steiner-cut-LP

min $\sum_{e\in E} c(e)x(e)$

s.t. $\sum_{e\in\delta(S)} x(e) \geq 1 \quad \forall S \subseteq V - s, S \cap R \neq \emptyset$

$0 \leq x(e) \leq 1 \quad \forall e \in E$

PC-Steiner-cut-LP

min $\sum_{e\in E} c(e)x(e) + \sum_{v\in V} \pi(v)z(v)$

s.t. $\sum_{e\in\delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$

$z(s) = 0$

$0 \leq z(v) \leq 1 \quad \forall v \in V$

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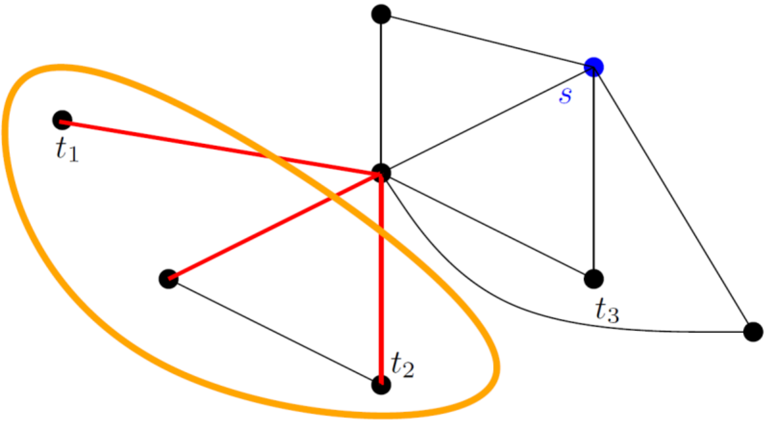

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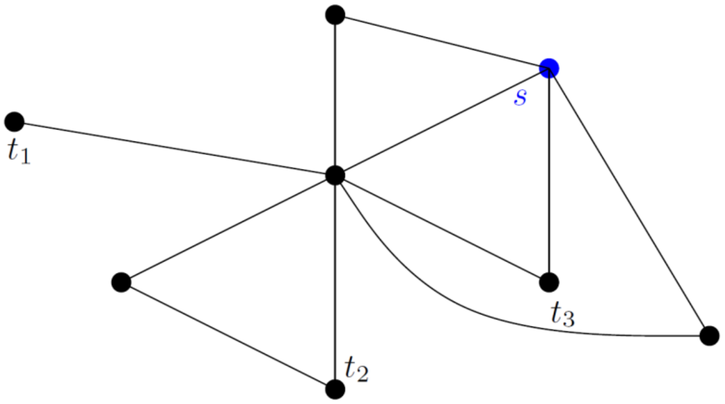

* Integrality gap of Steiner-cut-LP is 2

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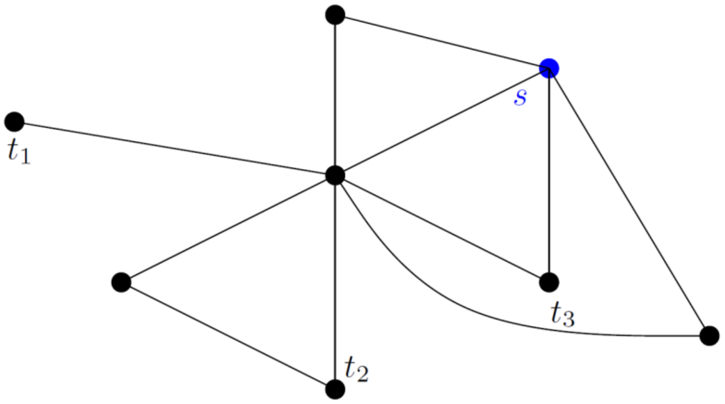

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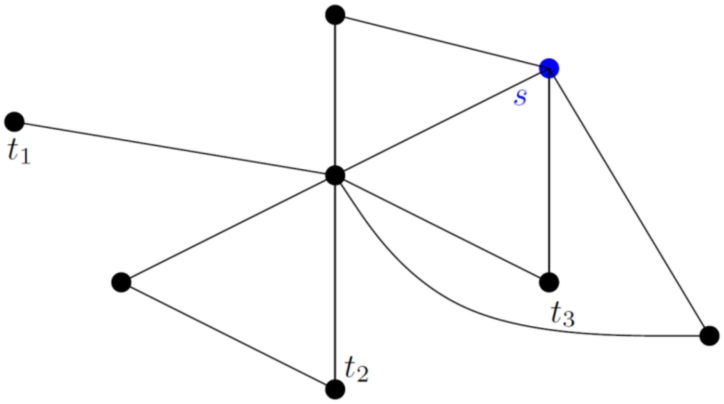
Rounding Method

PC–Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



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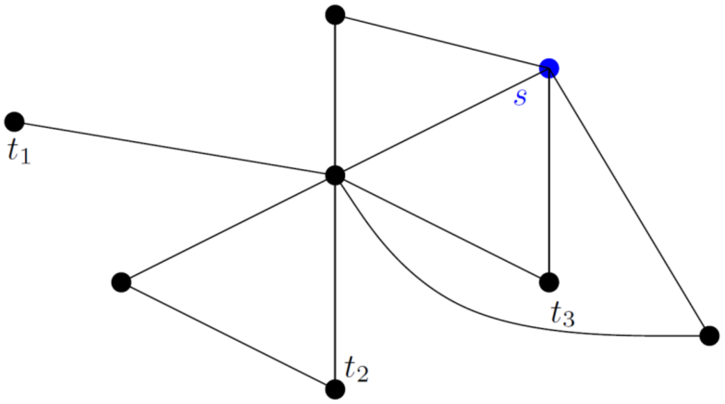

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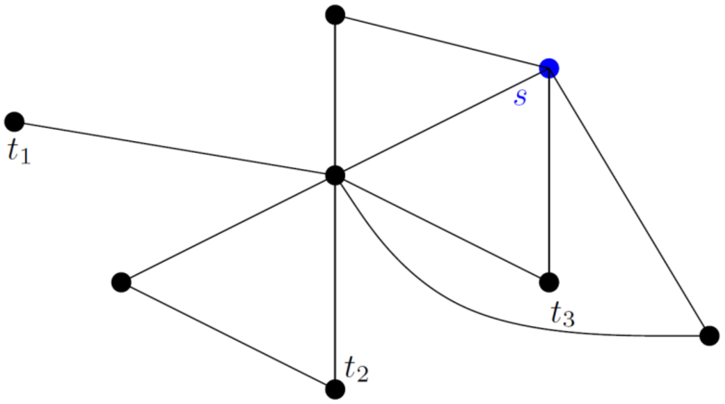
(x^*, z^*): Optimal solution to PC-Steiner-cut-LP

I : Set of all nodes such that $z(v) \geq 1/2$

Solve Steiner tree for the set of vertices in $\mathrm{J} = V - I$

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



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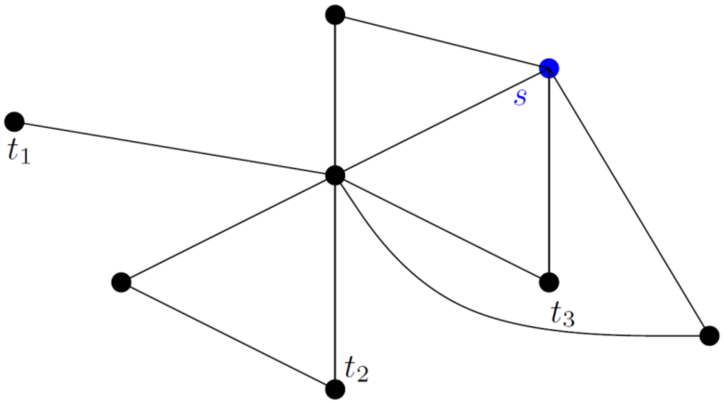
Solve Steiner tree for the set of vertices in J $= V - I$

(x^*, z^*) ⇨ Rounding z* ⇨ $z'(v) = 1$ iff $z^*(v) \ge \frac{1}{2}$ and 0 otw. ⇨ Scaling x* ⇨ $x'(e) = \min(1, 2x^*(e))$ ⇨ Feasible solution (x', z') with *integral* z values:

Val(x',z') ≤2Val(x*,z*)

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



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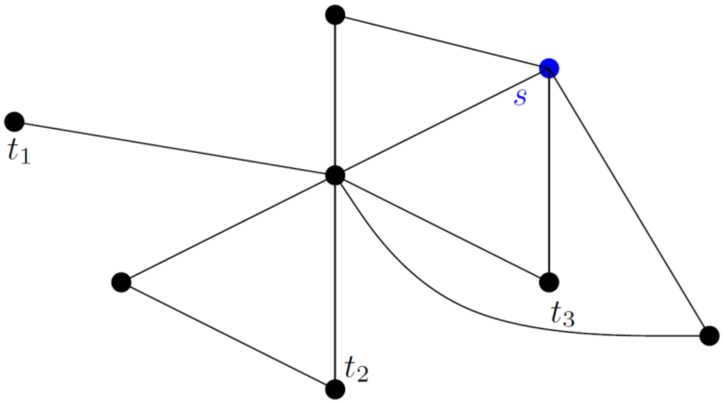
Solve Steiner tree for the set of vertices in J $= V - I$

(x^*, z^*) → Rounding z* → $z'(v) = 1$ iff $z^*(v) \geq \frac{1}{2}$ and 0 otw. → Scaling x* → $x'(v) = \min(1, 2x^*(v))$ → x' is a feasible solution of Steiner-cut-LP on node set J

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



PC-Steiner-cut-LP

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I : Set of all nodes such that $z(v) \geq 1/2$
Solve Steiner tree for the set of vertices in J $= V - I$

Integrality gap of Steiner-cut-LP is 2

T: 2-approximate solution of Steiner-cut-LP instance

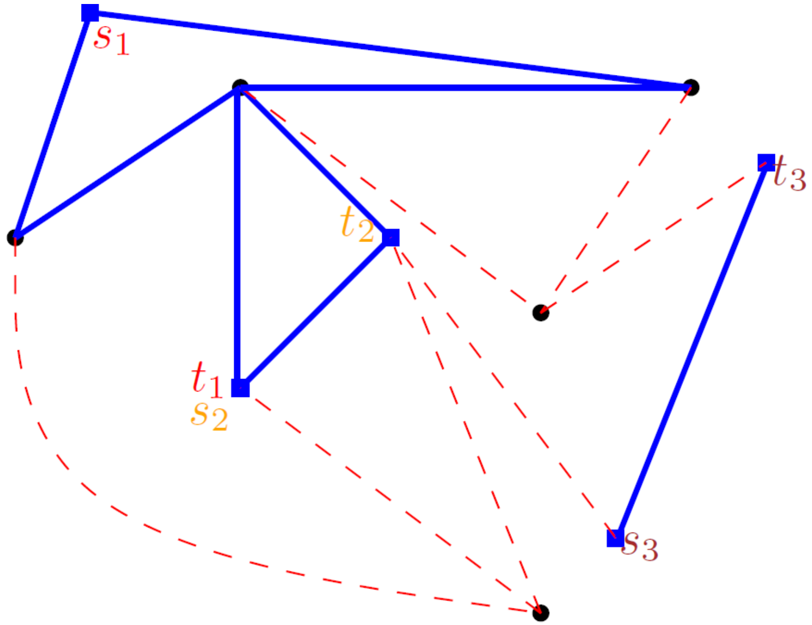
T is a 4-approximate solution of PC-Steiner-cut-LP

LP Relaxation for SNDP

For $k \geq 2$, no LP relaxations for node-weighted SNDP (and PC-SNDP) is known.
However, cut-LP works for node-weighted Steiner tree/forest

An LP relaxation for node-weighted SNDP in higher connectivity is required!

Multiroute-flow based LP for SNDP



Multiroute flow is considered in [Kishimoto '96] & [Aggrawal and Orlin '02]

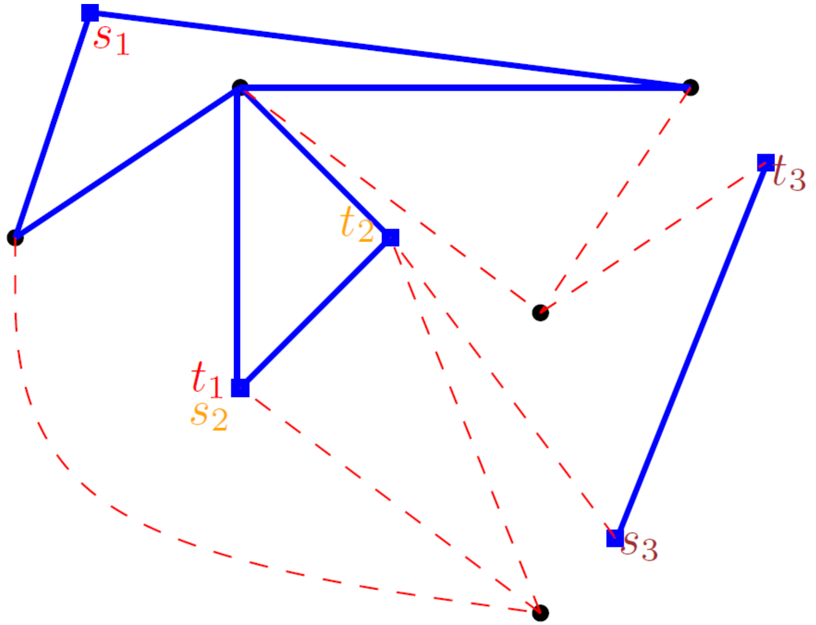
$\bar{p} = (p_1, \ldots, p_l)$: tuple of l disjoint st paths

$\mathcal{P}_{st}^{r(st)}$: Collection of all $r(st)$-tuples connecting s to t

$f(\bar{p}) = 1$ if the paths connecting s to t are the paths of $\bar{p}$

Connectivity constraint: $\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1$

Multiroute-flow based LP for SNDP



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Capacity Constraint

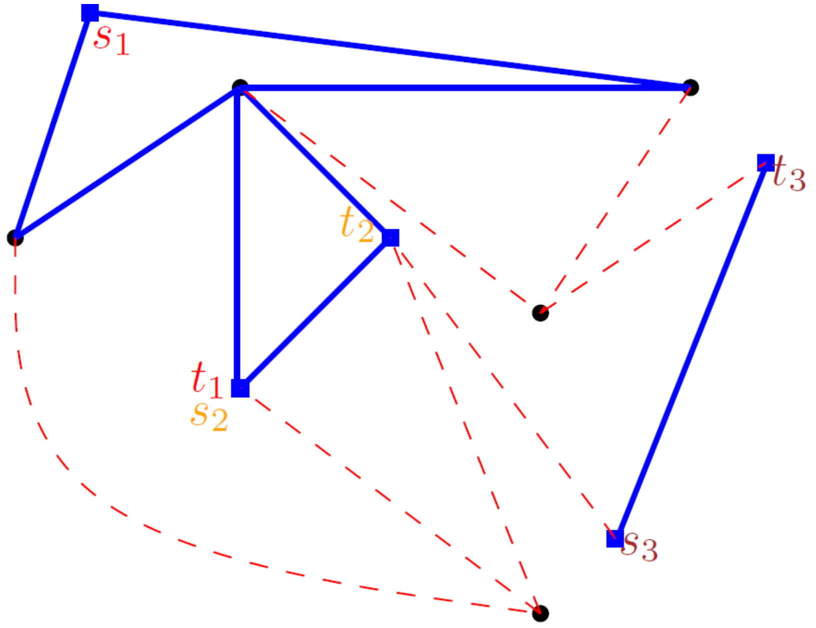
Edge weighted graphs

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}, e \in \bar{p}} f(\bar{p}) \leq x(e) \quad \forall e, \forall st$$

Node weighted graphs

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}, v \in \bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$$

Multiroute-flow based LP for Edge Weighted SNDP



Multiroute-LP

$$\min \sum_{e\in E} w(e)x(e)$$

$$\text{s.t.} \quad \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \qquad \forall st$$

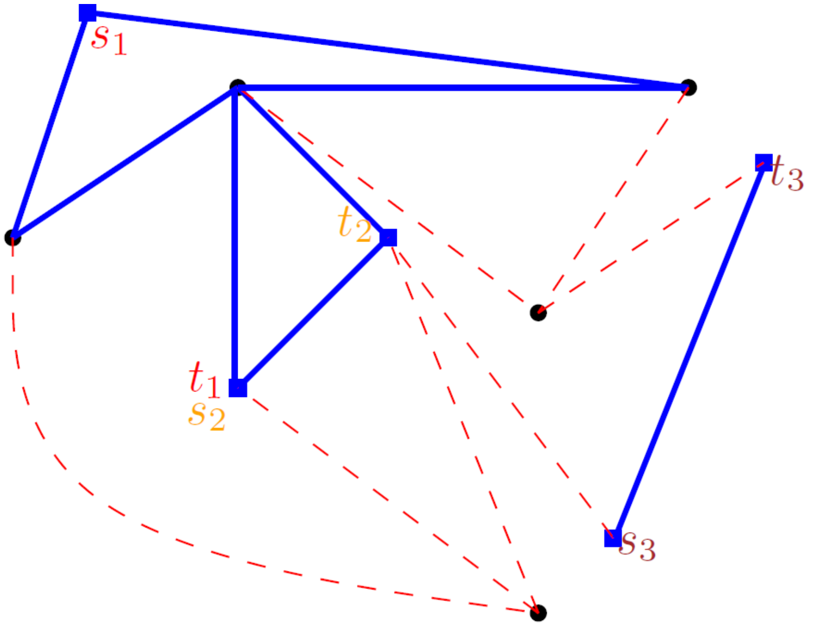
$$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, e\in\bar{p}} f(\bar{p}) \leq x(e) \qquad \forall e, \forall st$$

$$f(\bar{p}) \geq 0 \qquad \forall \bar{p}$$

- Separation oracle is min-cost flow

- It is equivalent to the cut-LP based relaxation with additional flow variables of [Hajiaghayi et al. '10]

Multiroute-flow based LP for Node Weighted SNDP



Multiroute-LP

min $\sum_{v \in V} w(v)x(v)$

s.t. $\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \qquad \forall st$

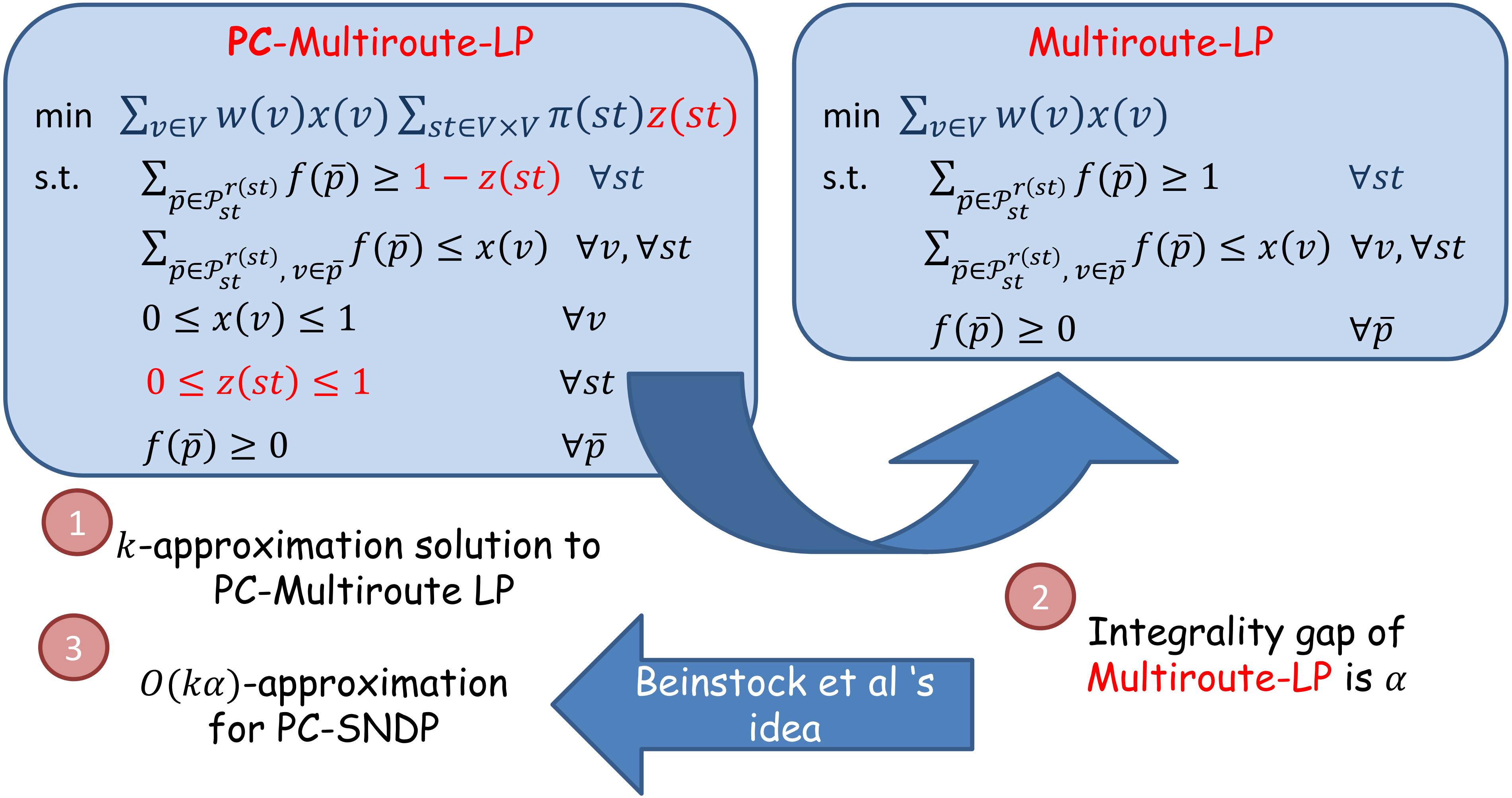
$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)},\, v \in \bar{p}} f(\bar{p}) \leq x(v) \qquad \forall v, \forall st$

$f(\bar{p}) \geq 0 \qquad \forall \bar{p}$

- There is no polynomial separation oracle
  - Even it is NP-hard for a single pair (s,t) to find k edge-disjoint paths in node-weighted graphs

We can find a k-approximate solution in polynomial by solving another LP-relaxation

Multiroute-flow based LP for Node Weighted PC-SNDP



PC-Multiroute-LP

$$
\begin{array}{lll}
\text{min} & \sum_{v\in V} w(v)x(v) \sum_{st\in V\times V} \pi(st)z(st) & \\
\text{s.t.} & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 - z(st) & \forall st \\
 & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) & \forall v, \forall st \\
 & 0 \leq x(v) \leq 1 & \forall v \\
 & 0 \leq z(st) \leq 1 & \forall st \\
 & f(\bar{p}) \geq 0 & \forall \bar{p}
\end{array}
$$

Multiroute-LP

$$
\begin{array}{lll}
\text{min} & \sum_{v\in V} w(v)x(v) & \\
\text{s.t.} & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 & \forall st \\
 & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) & \forall v, \forall st \\
 & f(\bar{p}) \geq 0 & \forall \bar{p}
\end{array}
$$

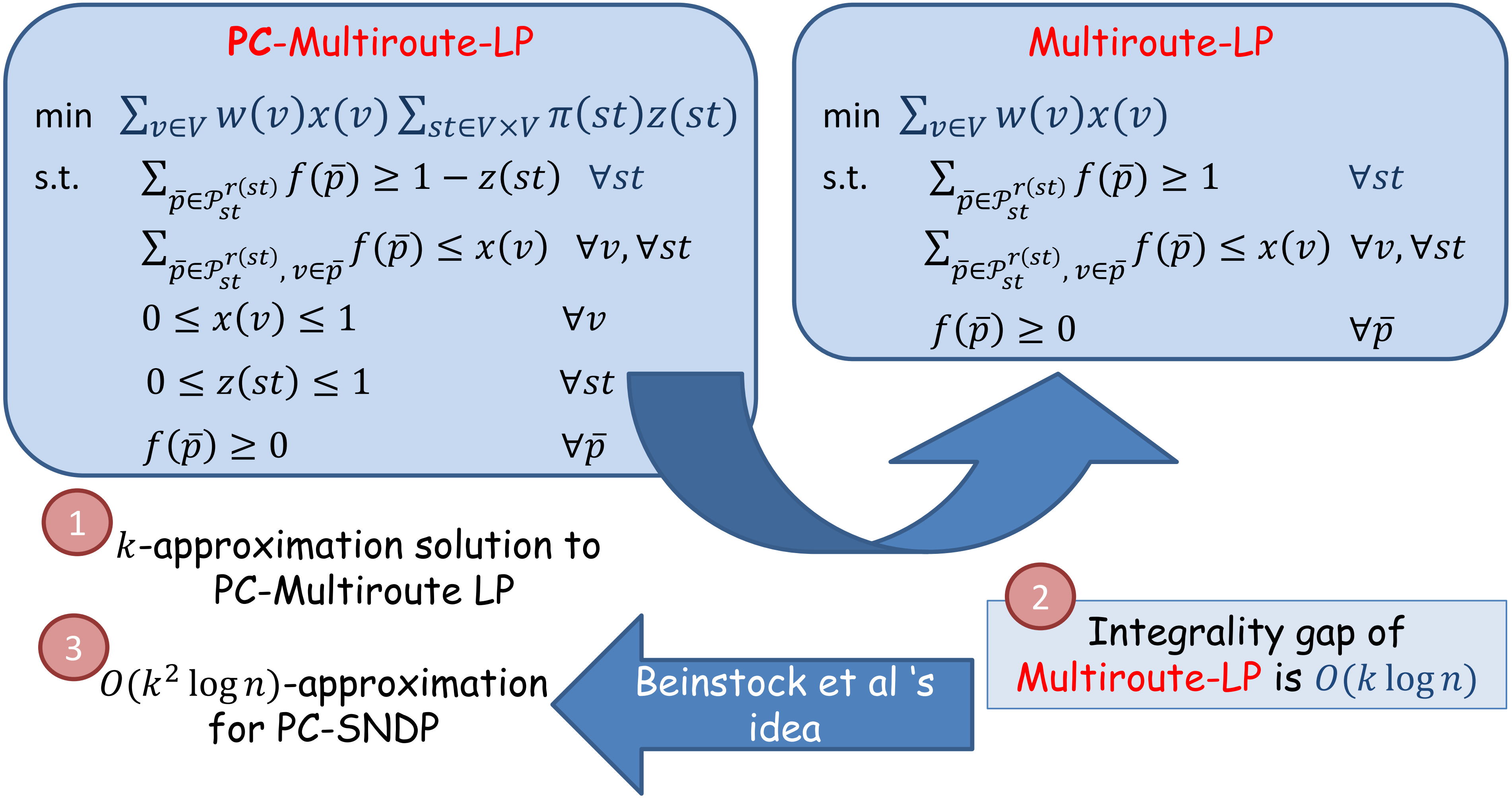
① k-approximation solution to PC-Multiroute LP

② Integrality gap of Multiroute-LP is α

③ $O(k\alpha)$-approximation for PC-SNDP

Beinstock et al 's idea

Multiroute-flow based LP for Node Weighted PC-SNDP



PC-Multiroute-LP

$$\begin{aligned} \min \quad & \sum_{v\in V} w(v)x(v) \sum_{st\in V\times V} \pi(st)z(st) \\ \text{s.t.} \quad & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 - z(st) && \forall st \\ & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) && \forall v, \forall st \\ & 0 \leq x(v) \leq 1 && \forall v \\ & 0 \leq z(st) \leq 1 && \forall st \\ & f(\bar{p}) \geq 0 && \forall \bar{p} \end{aligned}$$

Multiroute-LP

$$\begin{aligned} \min \quad & \sum_{v\in V} w(v)x(v) \\ \text{s.t.} \quad & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 && \forall st \\ & \sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) && \forall v, \forall st \\ & f(\bar{p}) \geq 0 && \forall \bar{p} \end{aligned}$$

① k-approximation solution to PC-Multiroute LP

② Integrality gap of Multiroute-LP is $O(k \log n)$

Beinstock et al 's idea

③ $O(k^2 \log n)$-approximation for PC-SNDP

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.

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  - We prove the same ratio by considering Augment-LP relaxation.

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In each phase:

$OPT(\text{AugmentLP}) \leq OPT(\text{MultirouteLP})$

Integrality gap of Multiroute-LP is $O(k \log n)$

Questions?

Thanks!

Different LP Relaxation

It is *not* based on Multiroute flow!
In a feasible solution H, s and t are r(s,t)-connected.

[Menger's theorem] By omitting $\ell \leq r(s,t)$ edges form H, s and t remain connected.

We can write an LP-relaxation for SNDP problem based on this property.

The exact optimal solution can be found.

However; its running time is polynomial in n^k