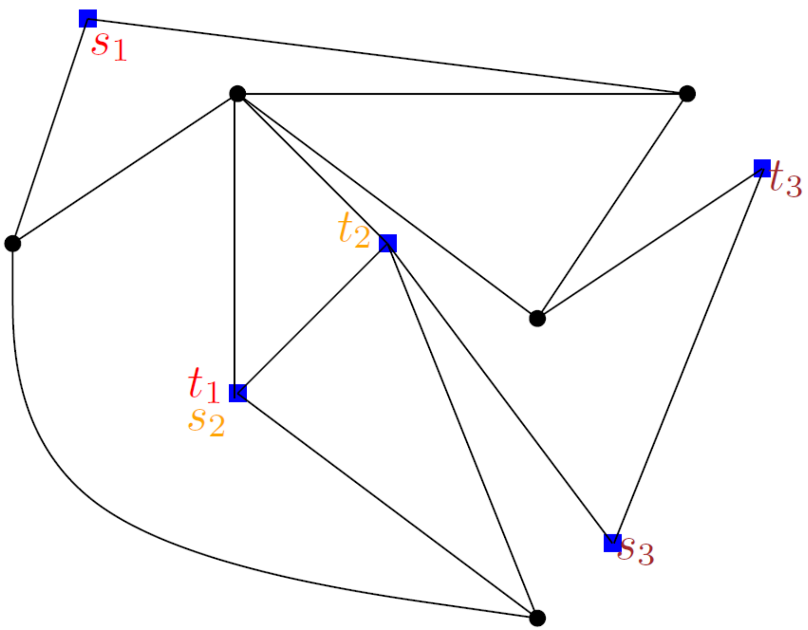
Prize-collecting Survivable Network Design in Node-weighted Graphs

Chandra Chekuri   Alina Ene   Ali Vakilian

ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Survivable Network Design (SNDP)



Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

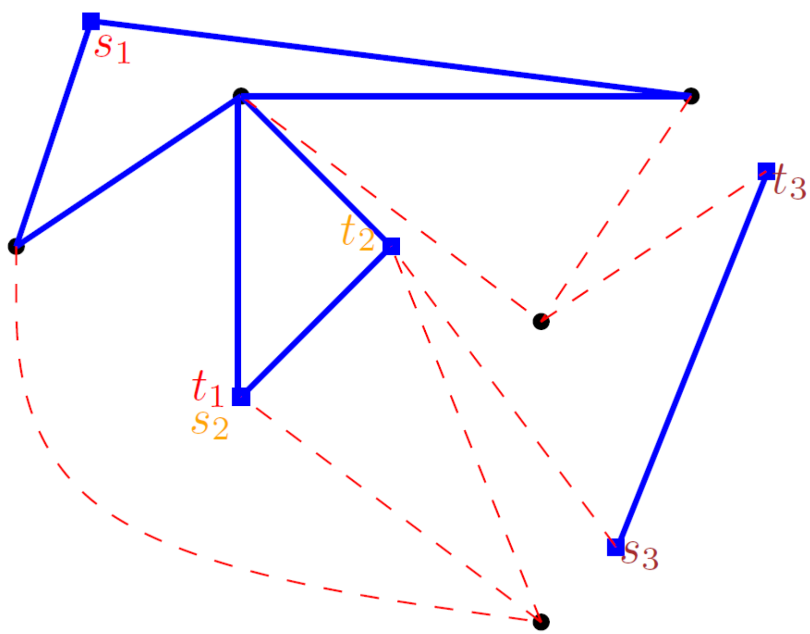
$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \le r} r(s_i, t_i)$

Goal: Min the sum of the weight of subgraph H containing $r(s_i, t_i)$ disjoint paths between s_i and t_i.

$$r(s_1, t_1) = 2$$
$$r(s_2, t_2) = 2$$
$$r(s_3, t_3) = 1$$

Survivable Network Design (SNDP)



Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

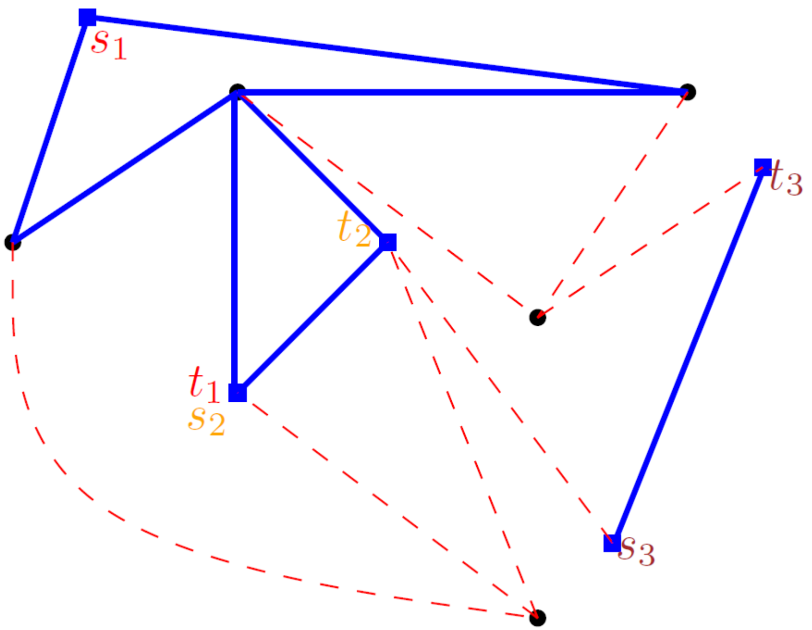
k = maximum requirement; $\max_{i \le r} r(s_i, t_i)$

Goal: Min the sum of the weight of subgraph H containing $r(s_i, t_i)$ disjoint paths between s_i and t_i.

$r(s_1, t_1) = 2$
$r(s_2, t_2) = 2$
$r(s_3, t_3) = 1$

Well-known special cases: Steiner tree/forest (k=1)

Survivable Network Design (SNDP)



Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

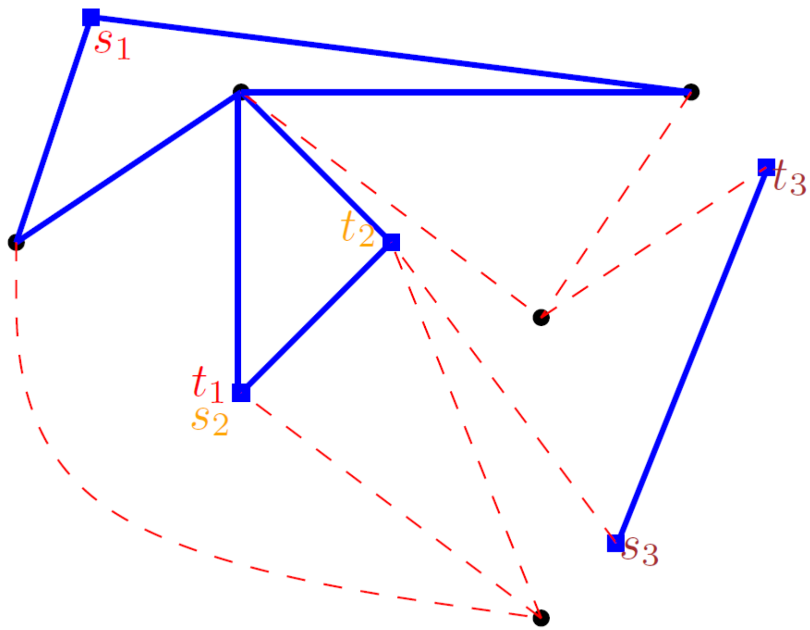
$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \le r} r(s_i, t_i)$

Goal: Min the sum of the weight of subgraph H containing $r(s_i, t_i)$ disjoint paths between s_i and t_i.

	Edge dt.	Element dt.	Vertex dt.
Edge wt.	2-approx Jain '98	2-approx Fleischer et al. '01	$O(k^3 \log n)$ Chuzhoy-Khanna '09
Node wt.	$\Omega(\log n)$-hard for Steiner tree Klein and Ravi '95		

Survivable Network Design (SNDP)



Collection of l pairs: $(s_1, t_1), \ldots, (s_l, t_l)$

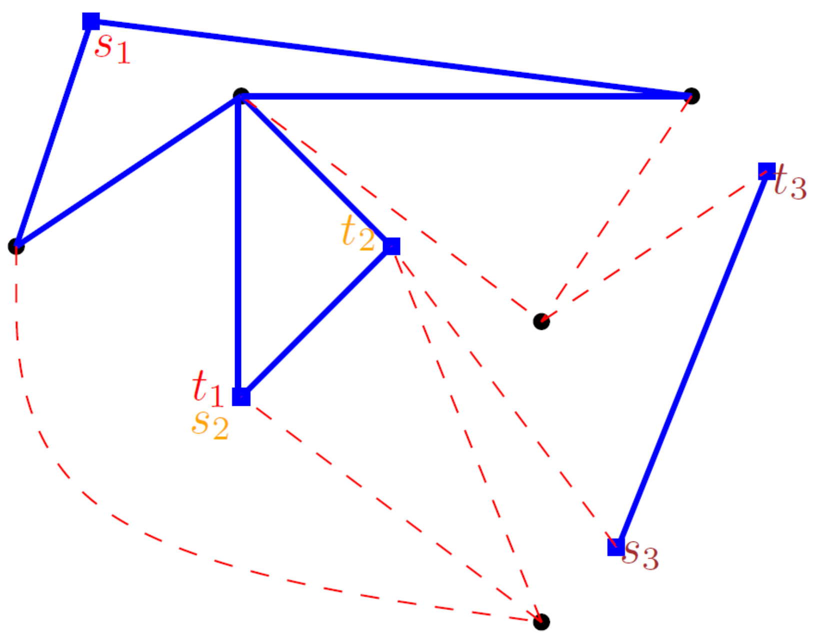
$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \le r} r(s_i, t_i)$

Goal: Min the sum of the weight of subgraph H containing $r(s_i, t_i)$ disjoint paths between s_i and t_i.

	Edge dt.	Element dt.	Vertex dt.
Edge wt.	2-approx Jain '98	2-approx Fleischer et al. '01	$O(k^3 \log n)$ Chuzhoy-Khanna '09
Node wt.	$O(k \log n)$ Nutov '08	$O(k \log n)$ Nutov '09	$O(k^4 \log^2 n)$ Nutov '09

Prize-collecting Survivable Network Design (PC-SNDP)



Collection of l pairs: $(s_1, t_1), \ldots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \le r} r(s_i, t_i)$

$\pi(s_i, t_i)$: penalty for not satisfying the connectivity of pair (s_i, t_i)

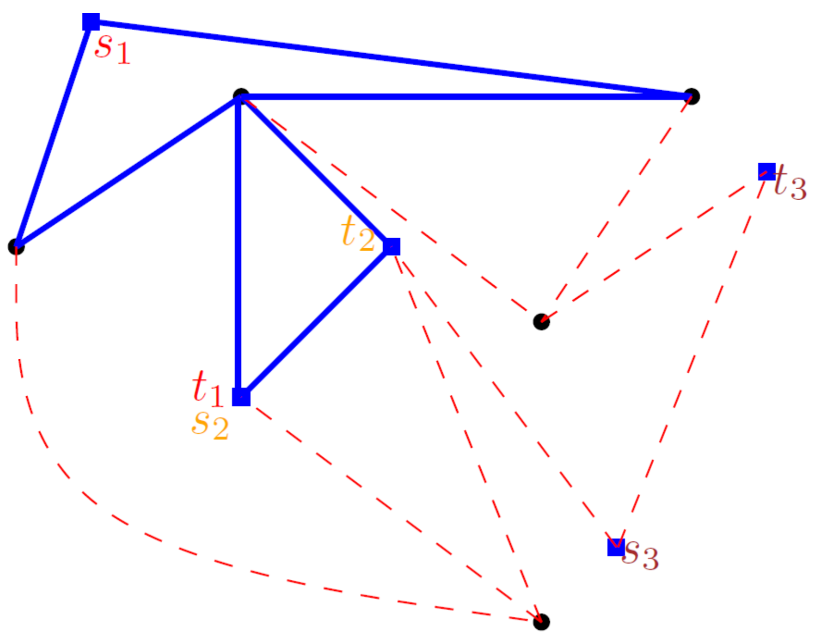
Goal: Min the sum of the weight of subgraph H + *the sum of penalties for requirements not satisfied by* H.

$r(s_1, t_1) = 2 \quad \pi(s_1, t_1) = 10$

$r(s_2, t_2) = 2 \quad \pi(s_2, t_2) = 20$

$r(s_3, t_3) = 1 \quad \pi(s_3, t_3) = 2$

Prize-collecting Survivable Network Design (PC-SNDP)



Collection of l pairs: $(s_1, t_1), \ldots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \le r} r(s_i, t_i)$

$\pi(s_i, t_i)$: penalty for not satisfying the connectivity of pair (s_i, t_i)

Goal: Min the sum of the weight of subgraph H + the sum of penalties for requirements not satisfied by H.

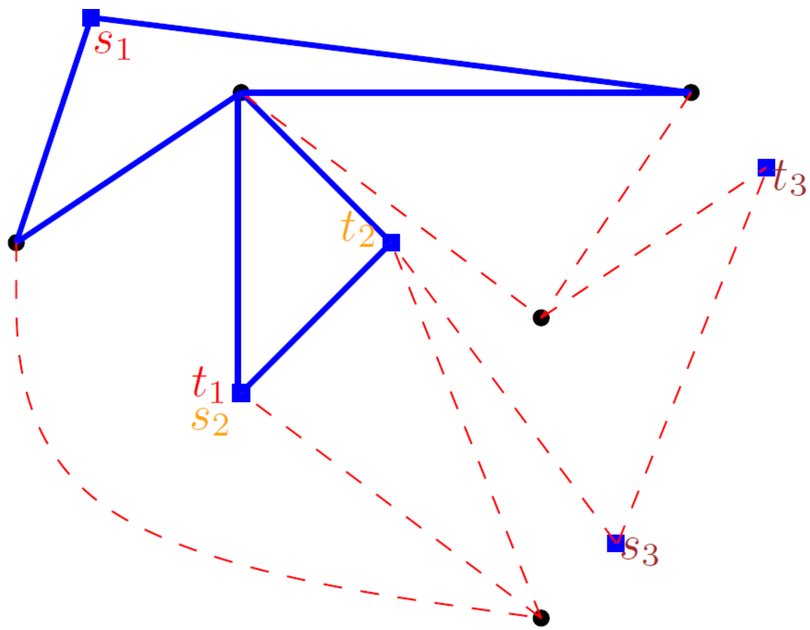
$r(s_1, t_1) = 2 \quad \pi(s_1, t_1) = 10$

$r(s_2, t_2) = 2 \quad \pi(s_2, t_2) = 20$

$r(s_3, t_3) = 1 \quad \pi(s_3, t_3) = 2$

All-or-nothing penalty version

Prize-collecting Survivable Network Design (PC-SNDP)



Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

k = maximum requirement; $\max_{i \leq r} r(s_i, t_i)$

$\pi(s_i, t_i)$: penalty for not satisfying the connectivity of pair (s_i, t_i)

Goal: Min the sum of the weight of subgraph H + the sum of penalties for requirements not satisfied by H.

	Edge dt.	Element dt.	Vertex dt.
Edge wt.	2.54-approx Hajiaghayi et al. '10	2.54-approx Hajiaghayi et al. '10	$O(k^3 \log n)$ Hajiaghayi et al. '10
Node wt.	No result	No result	No result

Our Result

First approximation for node weighted PC-SNDP

	Edge dt.	Element dt.	Vertex dt.
Node wt.	$O(k^2 \log n)$	$O(k^2 \log n)$	$O(k^5 \log^2 n)$
	$O(k \log n)$*	$O(k \log n)$*	$O(k^4 \log^2 n)$*

In planar graphs [Chekuri et al. '12]:

	Edge dt.	Element dt.	Vertex dt.
Node wt.	$O(k^2)$	$O(k^2)$	$O(k^5 \log n)$
	$O(k)$*	$O(k)$*	$O(k^4 \log n)$*

*Running time is polynomial in n^k

Multiroute-flow based LP relaxation for (PC-)SNDP

- No LP-relaxation for node-weighted SNDP was known before.

Our Result

First approximation for node weighted PC-SNDP

	Edge dt.	Element dt.	Vertex dt.
Node wt.	$O(k^2 \log n)$	$O(k^2 \log n)$	$O(k^5 \log^2 n)$
	$O(k \log n)$*	$O(k \log n)$*	$O(k^4 \log^2 n)$*

In planar graphs [Chekuri et al. '12]:

	Edge dt.	Element dt.	Vertex dt.
Node wt.	$O(k^2)$	$O(k^2)$	$O(k^5 \log n)$
	$O(k)$*	$O(k)$*	$O(k^4 \log n)$*

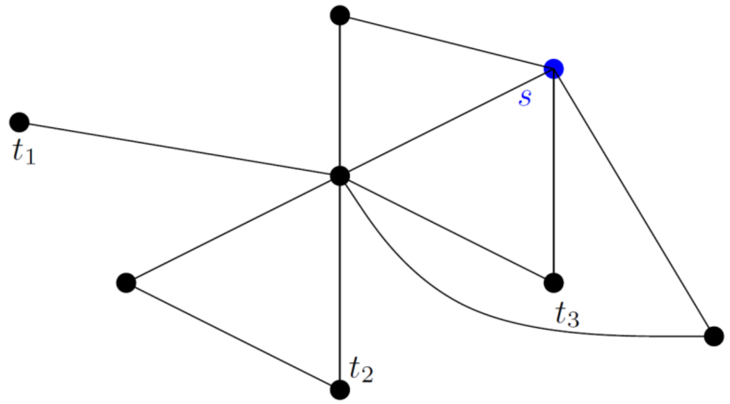
*Running time is polynomial in n^k

Multiroute-flow based LP relaxation for (PC-)SNDP

- No LP-relaxation for node-weighted SNDP was known before.

PC-Steiner Tree

Edge weighted, edge connectivity



R: Set of Steiner nodes

Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e)$$

$$\text{s.t. } \sum_{e \in \delta(S)} x(e) \geq 1 \quad \forall S \subseteq V - s, S \cap R \neq \emptyset$$

$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

PC-Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e) + \sum_{v \in V} \pi(v)z(v)$$

$$\text{s.t. } \sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$$

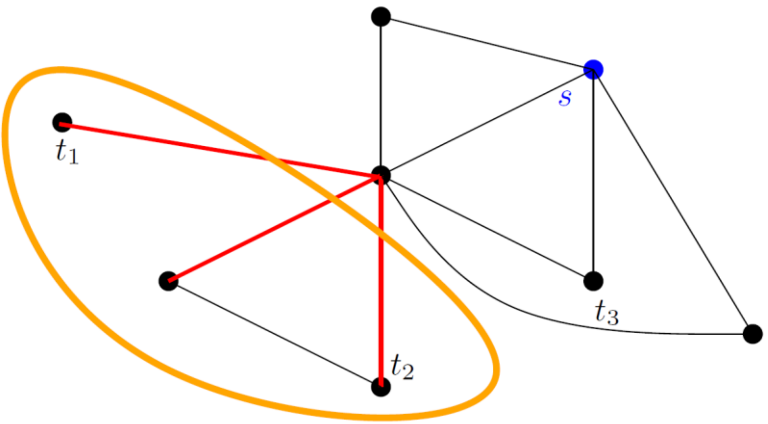
$$z(s) = 0$$

$$0 \leq z(v) \leq 1 \quad \forall v \in V$$

$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

PC-Steiner Tree

Edge weighted, edge connectivity



R: Set of Steiner nodes

Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e)$$

$$\text{s.t. } \sum_{e \in \delta(S)} x(e) \geq 1 \quad \forall S \subseteq V - s, S \cap R \neq \emptyset$$

$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

PC-Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e) + \sum_{v \in V} \pi(v)z(v)$$

$$\text{s.t. } \sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$$

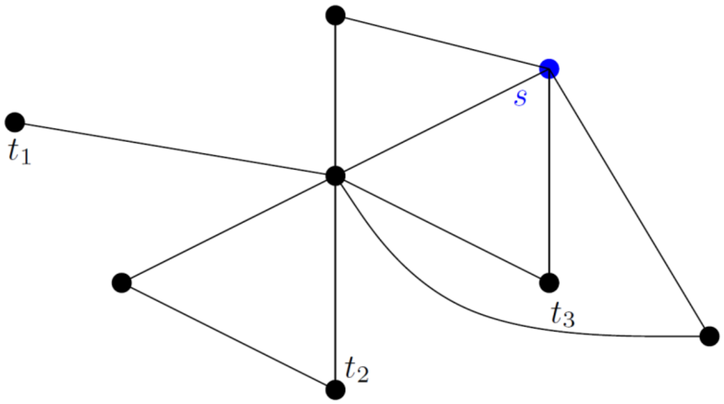
$$z(s) = 0$$

$$0 \leq z(v) \leq 1 \quad \forall v \in V$$

$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

PC-Steiner Tree

Edge weighted, edge connectivity



Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e)$$
$$\text{s.t.} \sum_{e \in \delta(S)} x(e) \geq 1 \quad \forall S \subseteq V - s, S \cap R \neq \emptyset$$
$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

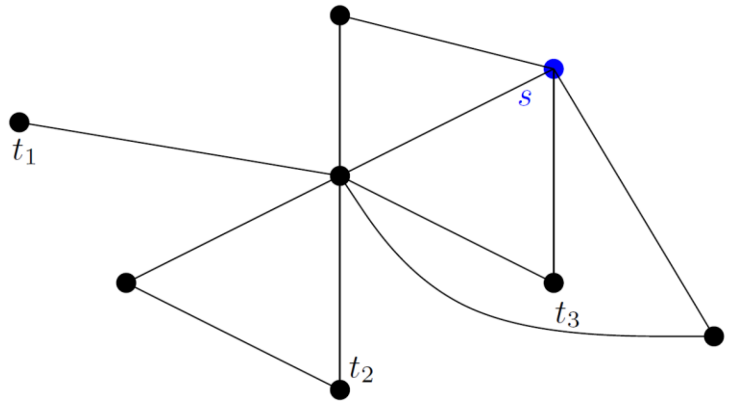
PC-Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e) + \sum_{v \in V} \pi(v)z(v)$$
$$\text{s.t.} \sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$$
$$z(s) = 0$$
$$0 \leq z(v) \leq 1 \quad \forall v \in V$$
$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

* Integrality gap of Steiner-cut-LP is 2

PC-Steiner Tree

Edge weighted, edge connectivity



Steiner-cut-LP

min $\sum_{e \in E} c(e)x(e)$

s.t. $\sum_{e \in \delta(S)} x(e) \geq 1 \quad \forall S \subseteq V - s, S \cap R \neq \emptyset$

$0 \leq x(e) \leq 1 \quad \forall e \in E$

PC-Steiner-cut-LP

min $\sum_{e \in E} c(e)x(e) + \sum_{v \in V} \pi(v)z(v)$

s.t. $\sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$

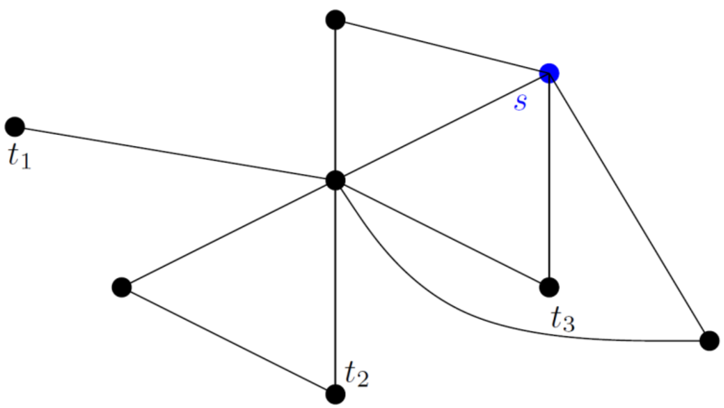
$z(s) = 0$

$0 \leq z(v) \leq 1 \quad \forall v \in V$

$0 \leq x(e) \leq 1 \quad \forall e \in E$

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



PC-Steiner-cut-LP

min $\sum_{e\in E} c(e)x(e) \; + \; \sum_{v\in V} \pi(v)z(v)$

s.t. $\sum_{e\in\delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$

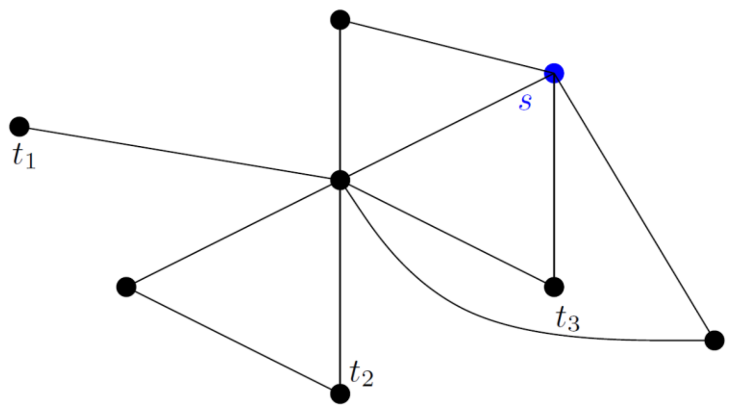
$z(s) = 0$

$0 \leq z(v) \leq 1 \quad \forall v \in V$

$0 \leq x(e) \leq 1 \quad \forall e \in E$

Rounding Method

PC–Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



PC-Steiner-cut-LP

$$\min \sum_{e \in E} c(e)x(e) + \sum_{v \in V} \pi(v)z(v)$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - s, \forall v \in S$$

$$z(s) = 0$$

$$0 \leq z(v) \leq 1 \quad \forall v \in V$$

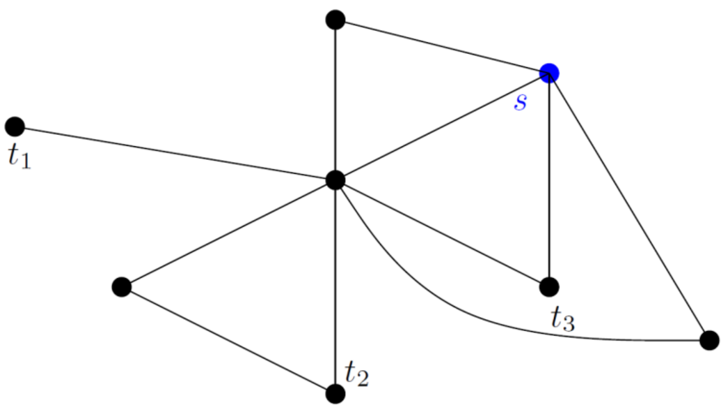
$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

(x^*, z^*): Optimal solution to PC-Steiner-cut-LP

I : Set of all nodes such that $z(v) \geq 1/2$

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



PC-Steiner-cut-LP

min $\sum_{e \in E} c(e)x(e) \; + \; \sum_{v \in V} \pi(v)z(v)$

s.t. $\sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - r, \forall v \in S$

$z(s) = 0$

$0 \leq z(v) \leq 1 \qquad \forall v \in V$

$0 \leq x(e) \leq 1 \qquad \forall e \in E$

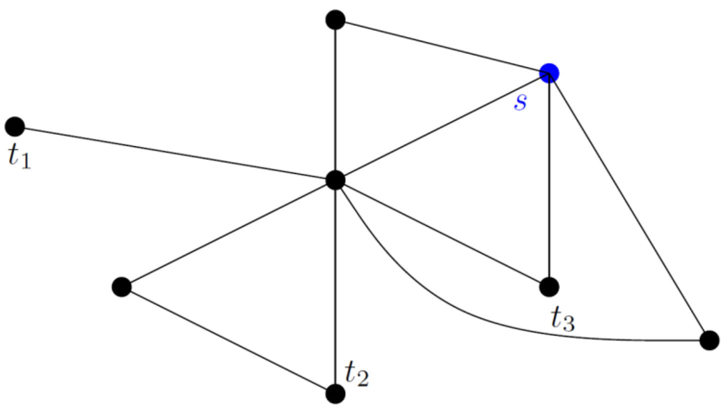
(x^*, z^*): Optimal solution to PC-Steiner-cut-LP

I : Set of all nodes such that $z(v) \geq 1/2$

(x^*, z^*) → Rounding z^* → $z'(v) = 1$ iff $z^*(v) \geq \frac{1}{2}$ and 0 otw. → Scaling x^* → $x'(v) = \min(1, 2x^*(v))$ → Feasible solution (x', z') with *integral* z values: Val(x',z') ≤2Val(x*,z*)

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



PC-Steiner-cut-LP

min $\sum_{e\in E} c(e)x(e) + \sum_{v\in V} \pi(v)z(v)$

s.t. $\sum_{e\in\delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - r, \forall v \in S$

$z(s) = 0$

$0 \leq z(v) \leq 1 \quad \forall v \in V$

$0 \leq x(e) \leq 1 \quad \forall e \in E$

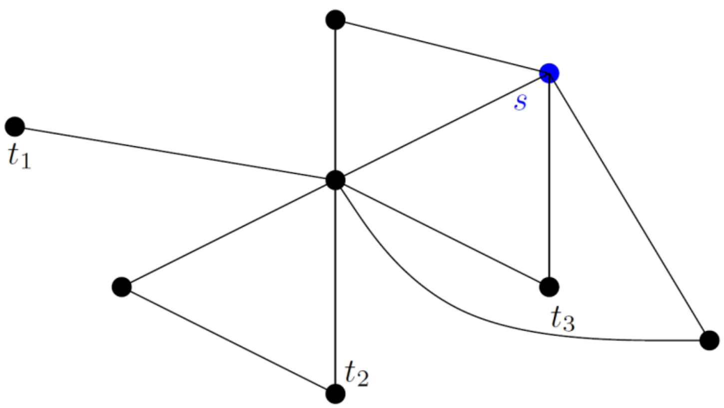
(x^*, z^*): Optimal solution to PC-Steiner-cut-LP

I : Set of all nodes such that $z(v) \geq 1/2$

(x^*, z^*) → Rounding z^* → $z'(v) = 1$ iff $z^*(v) \geq \frac{1}{2}$ and 0 otw. → Scaling x^* → $x'(v) = \min(1, 2x^*(v))$ → x' is a feasible solution of Steiner-cut-LP on node set J

Rounding Method

PC-Steiner Tree (edge weighted, edge connectivity) [Beinstock et al. '93]



PC-Steiner-cut-LP

min $\sum_{e \in E} c(e)x(e) + \sum_{v \in V} \pi(v)z(v)$

s.t. $\sum_{e \in \delta(S)} x(e) \geq 1 - z(v) \quad \forall S \subseteq V - r, \forall v \in S$

$z(s) = 0$

$0 \leq z(v) \leq 1 \quad \forall v \in V$

$0 \leq x(e) \leq 1 \quad \forall e \in E$

(x^*, z^*): Optimal solution to PC-Steiner-cut-LP
I : Set of all nodes such that $z(v) \geq 1/2$
Solve Steiner tree for the set of vertices in $J = V - I$

Integrality gap of Steiner-cut-LP is 2 → T: 2-approximate solution of Steiner-cut-LP instance

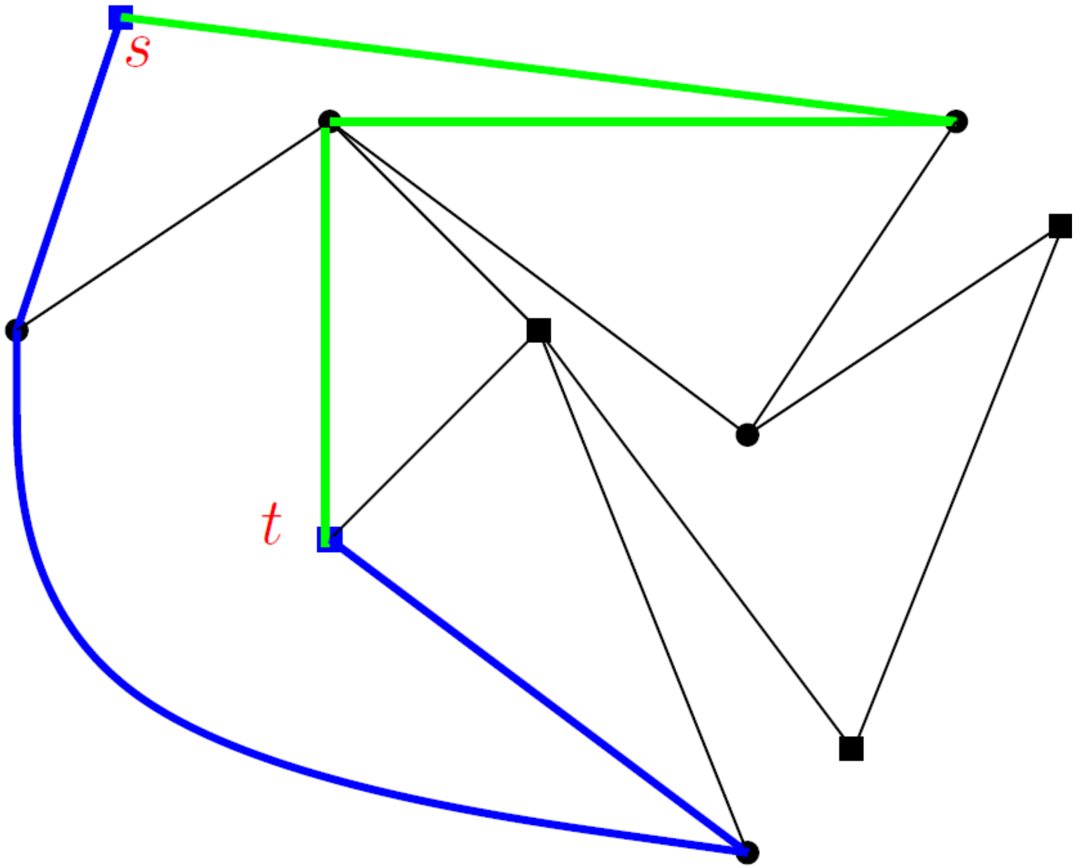
T is a 4-approximate solution of PC-Steiner-cut-LP

LP Relaxation for SNDP

For $k \geq 2$, no LP relaxations for node-weighted SNDP (and PC-SNDP) is known.
However, cut-LP works for node-weighted Steiner tree/forest

An LP relaxation for node-weighted SNDP in higher connectivity is required!

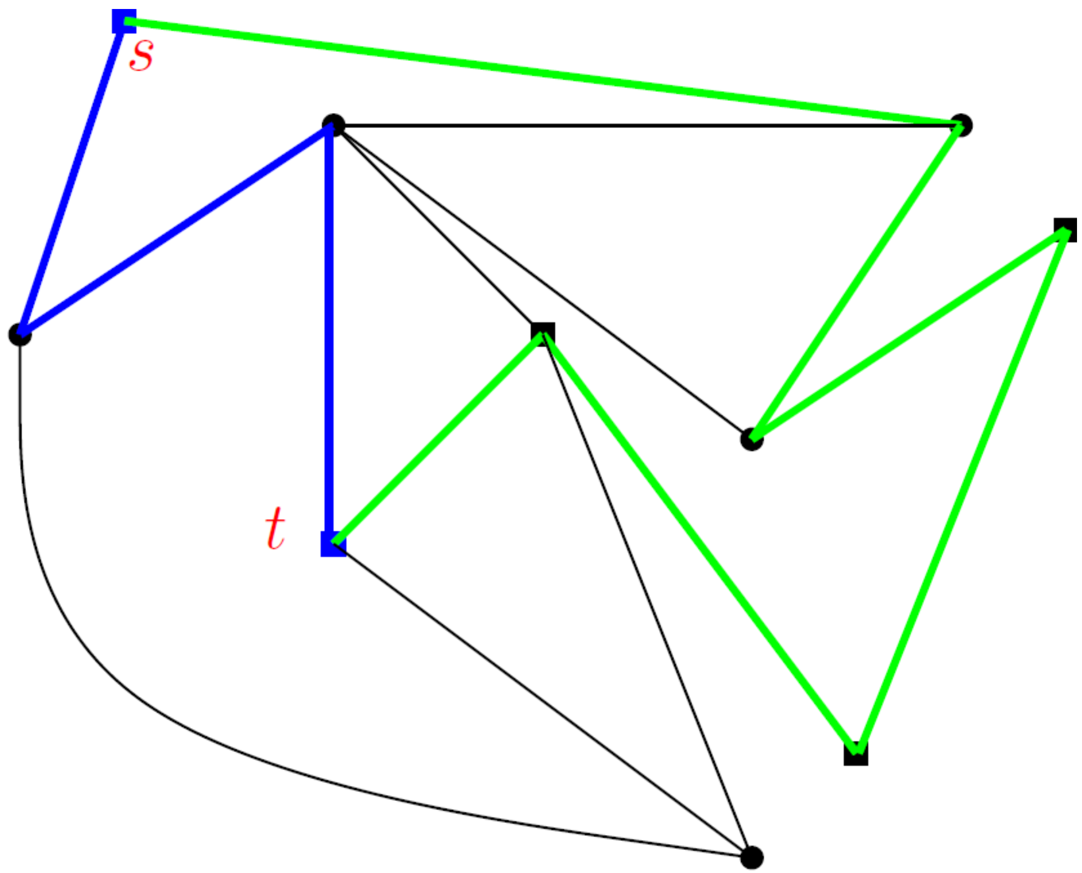
2-route-flow



Two disjoint paths Between s and t

Capacity of all edges are 1

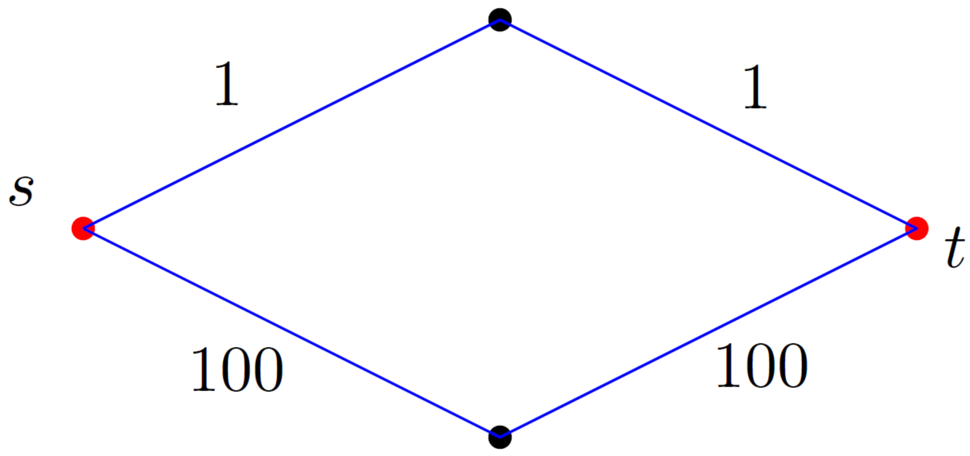
2-route-flow



Two disjoint paths Between s and t

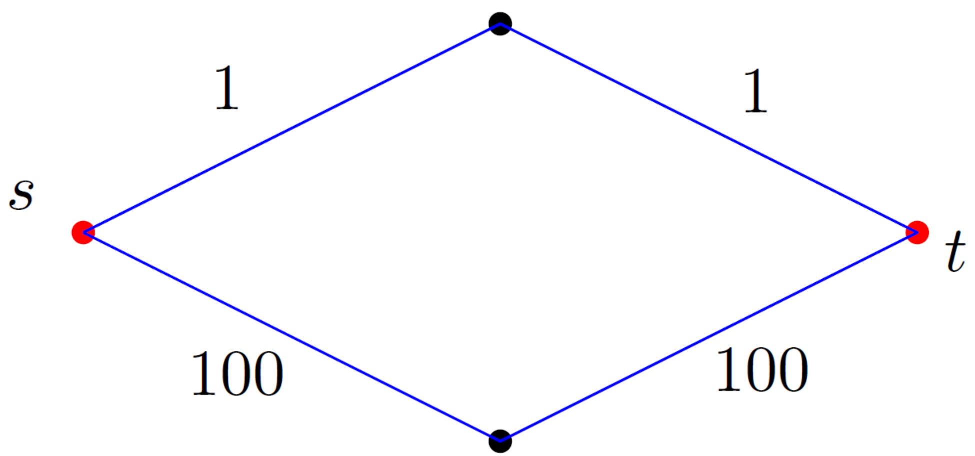
Capacity of all edges are 1

2-route-flow



Max flow is 101;
however,
max 2-route flow is 1.

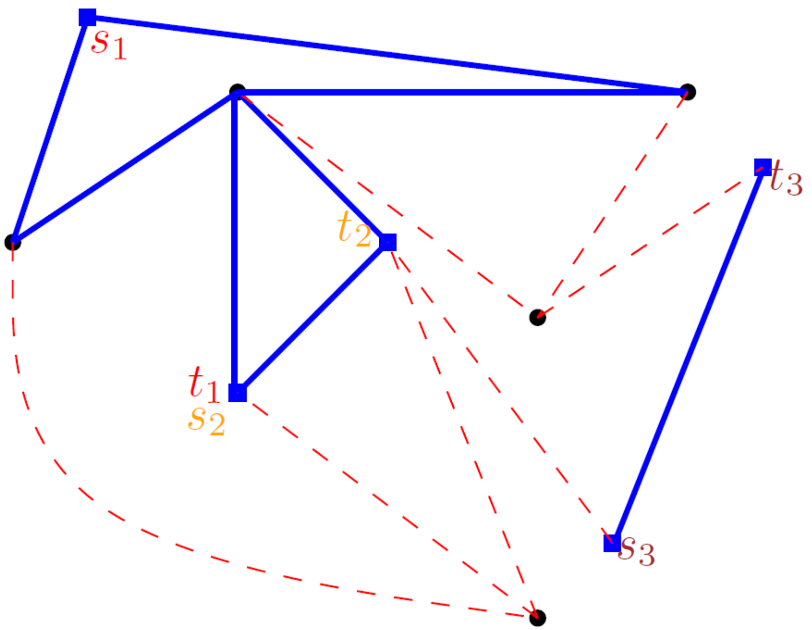
2-route-flow



Max flow is 101;
however,
max 2-route flow is 1.

If s and t are k-connected then k-route st-flow is at least 1.

Multiroute-flow based LP for SNDP



Multiroute flow is considered in [Kishimoto '96] & [Aggrawal and Orlin '02]

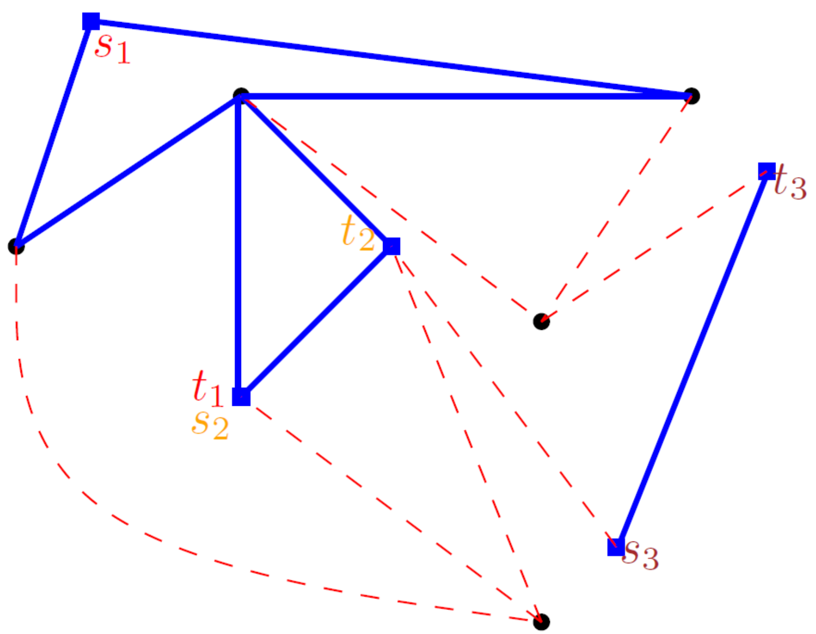
$\bar{p} = (p_1, \ldots, p_l)$: tuple of l disjoint st paths

$\mathcal{P}_{st}^{r(st)}$: Collection of all $r(st)$-tuples connecting s to t

$f(\bar{p}) = 1$ if the paths connecting s to t are the paths of $\bar{p}$; we have flow of value $r(st)$

Connectivity constraint: $\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1$

Multiroute-flow based LP for SNDP



Multiroute flow is considered in [Kishimoto '96] & [Aggrawal and Orlin '02]

$\bar{p} = (p_1, \dots, p_l)$: tuple of l disjoint st paths

$\mathcal{P}_{st}^{r(st)}$: Collection of all $r(st)$-tuples connecting s to t

$f(\bar{p}) = 1$ if the paths connecting s to t are the paths of $\bar{p}$; we have flow of value $r(st)$

Connectivity constraint: $\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1$

Capacity Constraint

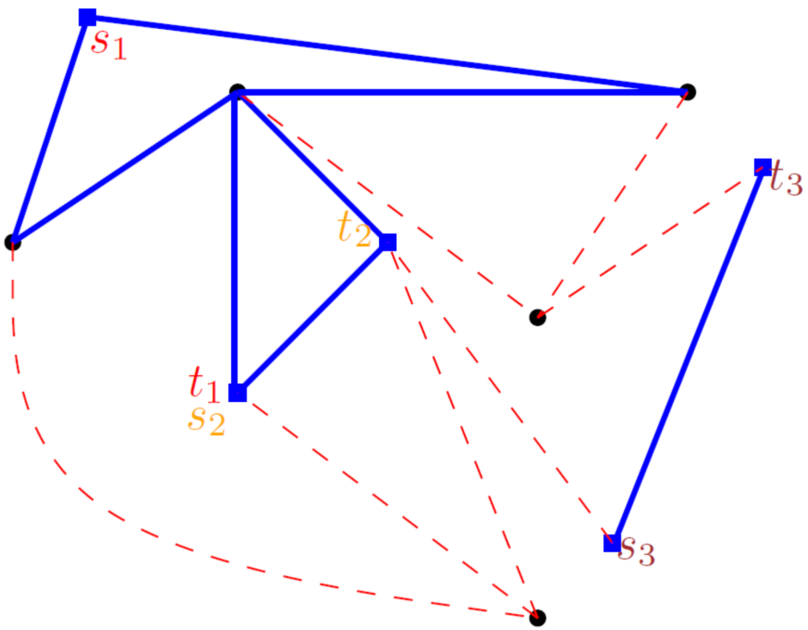
Edge weighted graphs

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}, e \in \bar{p}} f(\bar{p}) \leq x(e) \quad \forall e, \forall st$$

Node weighted graphs

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}, v \in \bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$$

Multiroute-flow based LP for Edge Weighted SNDP



Multiroute-LP

$$\min \sum_{e \in E} w(e) x(e)$$

$$\text{s.t.} \quad \sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \qquad \forall st$$

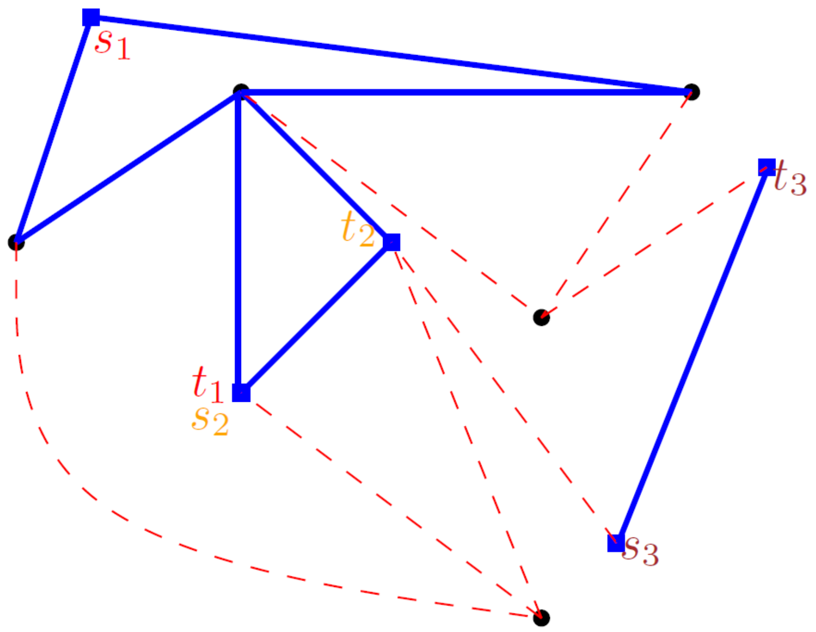
$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)},\, e \in \bar{p}} f(\bar{p}) \leq x(e) \qquad \forall e, \forall st$$

$$f(\bar{p}) \geq 0 \qquad \forall \bar{p}$$

- Separation oracle is min-cost flow

- It is equivalent to the cut-LP based relaxation with additional flow variables of [Hajiaghayi et al. '10]

Multiroute-flow based LP for Node Weighted SNDP



Multiroute-LP

min $\sum_{v \in V} w(v)x(v)$

s.t. $\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \quad \forall st$

$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)},\, v \in \bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$

$f(\bar{p}) \geq 0 \quad \forall \bar{p}$

- There is no polynomial separation oracle
  - Even it is NP-hard for a single pair (s,t) to find k edge-disjoint paths in node-weighted graphs (via set-cover)

We can find a k-approximate solution in polynomial by solving another LP-relaxation

Multiroute-flow based LP for Node Weighted PC-SNDP

PC-Multiroute-LP

$$\min \sum_{v \in V} w(v)x(v) \sum_{st \in V \times V} \pi(st)z(st)$$

s.t.

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 - z(st) \quad \forall st$$

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)},\, v \in \bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$$

$$0 \leq x(v) \leq 1 \quad \forall v$$

$$0 \leq z(st) \leq 1 \quad \forall st$$

$$f(\bar{p}) \geq 0 \quad \forall \bar{p}$$

Multiroute-LP

$$\min \sum_{v \in V} w(v)x(v)$$

s.t.

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \quad \forall st$$

$$\sum_{\bar{p} \in \mathcal{P}_{st}^{r(st)},\, v \in \bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$$

$$f(\bar{p}) \geq 0 \quad \forall \bar{p}$$

Multiroute-flow based LP for Node Weighted PC-SNDP

PC-Multiroute-LP

min $\sum_{v\in V} w(v)x(v) \sum_{st\in V\times V} \pi(st)z(st)$

s.t.

$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 - z(st) \quad \forall st$

$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$

$0 \leq x(v) \leq 1 \quad \forall v$

$0 \leq z(st) \leq 1 \quad \forall st$

$f(\bar{p}) \geq 0 \quad \forall \bar{p}$

Multiroute-LP

min $\sum_{v\in V} w(v)x(v)$

s.t.

$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \quad \forall st$

$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$

$f(\bar{p}) \geq 0 \quad \forall \bar{p}$

1. k-approximation solution to PC-Multiroute LP
2. Integrality gap of Multiroute-LP is α
3. $O(k\alpha)$-approximation for PC-SNDP

Beinstock et al 's method

Multiroute-flow based LP for Node Weighted PC-SNDP

PC-Multiroute-LP

min $\sum_{v\in V} w(v)x(v) \sum_{st\in V\times V} \pi(st)z(st)$

s.t. $\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 - z(st) \quad \forall st$

$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$

$0 \leq x(v) \leq 1 \quad \forall v$

$0 \leq z(st) \leq 1 \quad \forall st$

$f(\bar{p}) \geq 0 \quad \forall \bar{p}$

Multiroute-LP

min $\sum_{v\in V} w(v)x(v)$

s.t. $\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)}} f(\bar{p}) \geq 1 \quad \forall st$

$\sum_{\bar{p}\in\mathcal{P}_{st}^{r(st)},\, v\in\bar{p}} f(\bar{p}) \leq x(v) \quad \forall v, \forall st$

$f(\bar{p}) \geq 0 \quad \forall \bar{p}$

1. k-approximation solution to PC-Multiroute LP
2. Integrality gap of Multiroute-LP is $O(k \log n)$
3. $O(k^2 \log n)$-approximation for PC-SNDP

Beinstock et al 's method

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

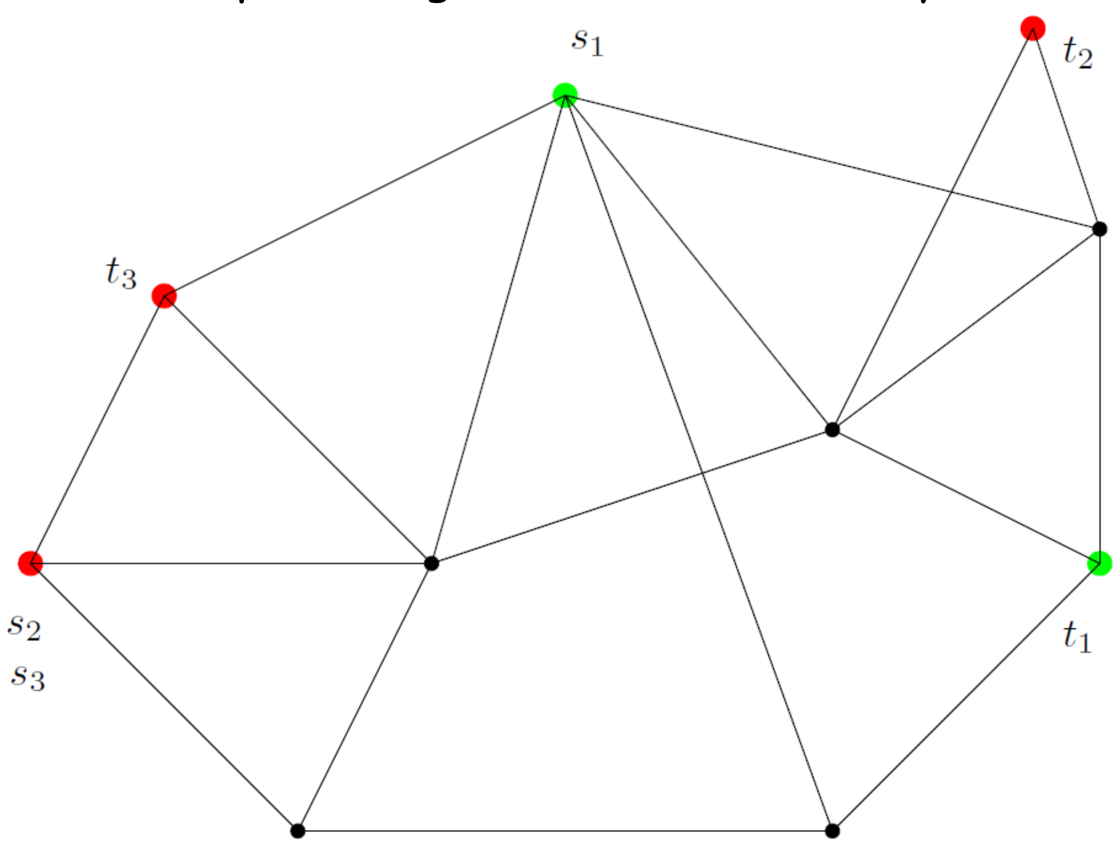
- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.



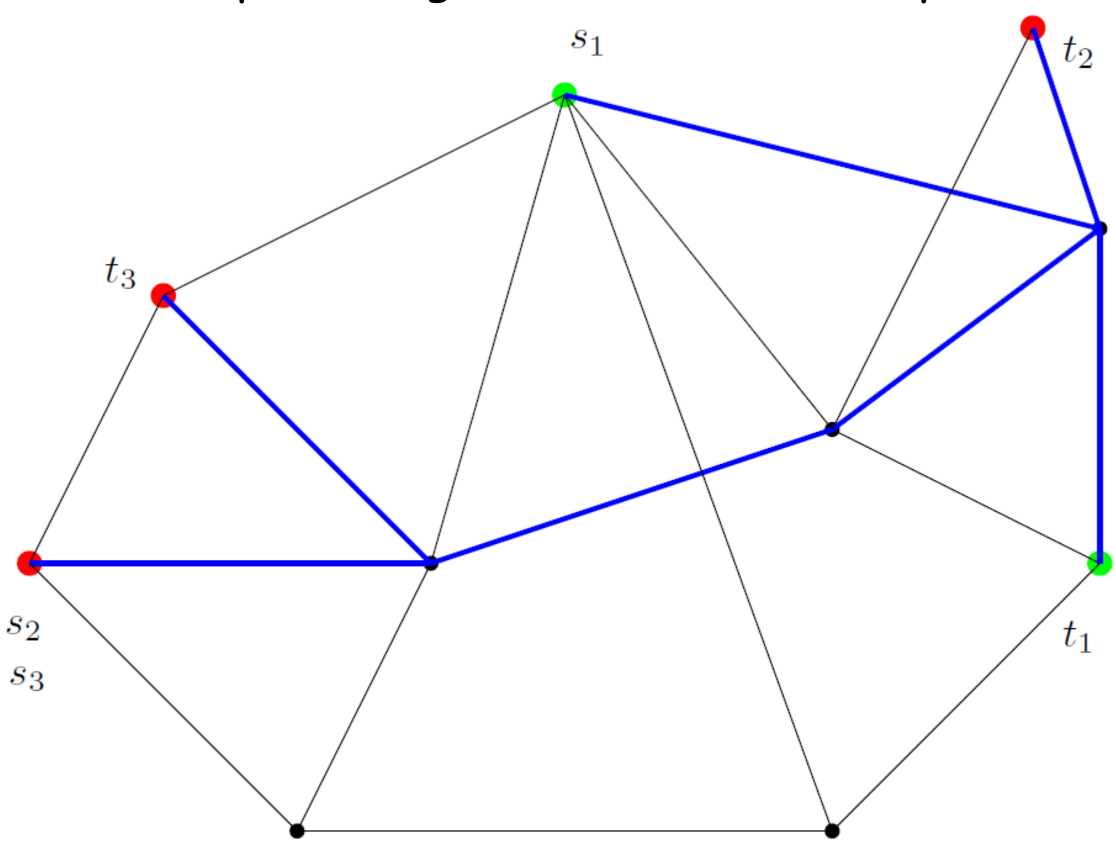
r($s_1 t_1$) = 3
r($s_2 t_2$) = 2
r($s_3 t_3$) = 2

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.



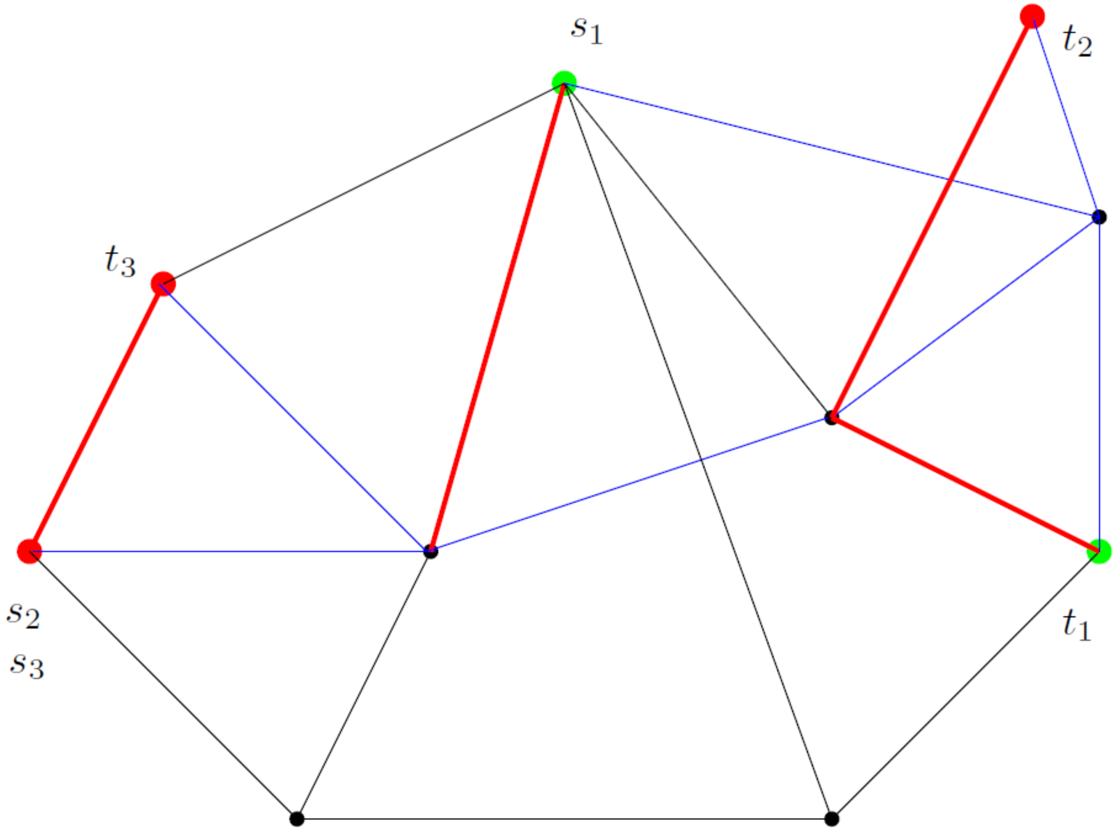
r($s_1 t_1$) = 3
r($s_2 t_2$) = 2
r($s_3 t_3$) = 2

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.



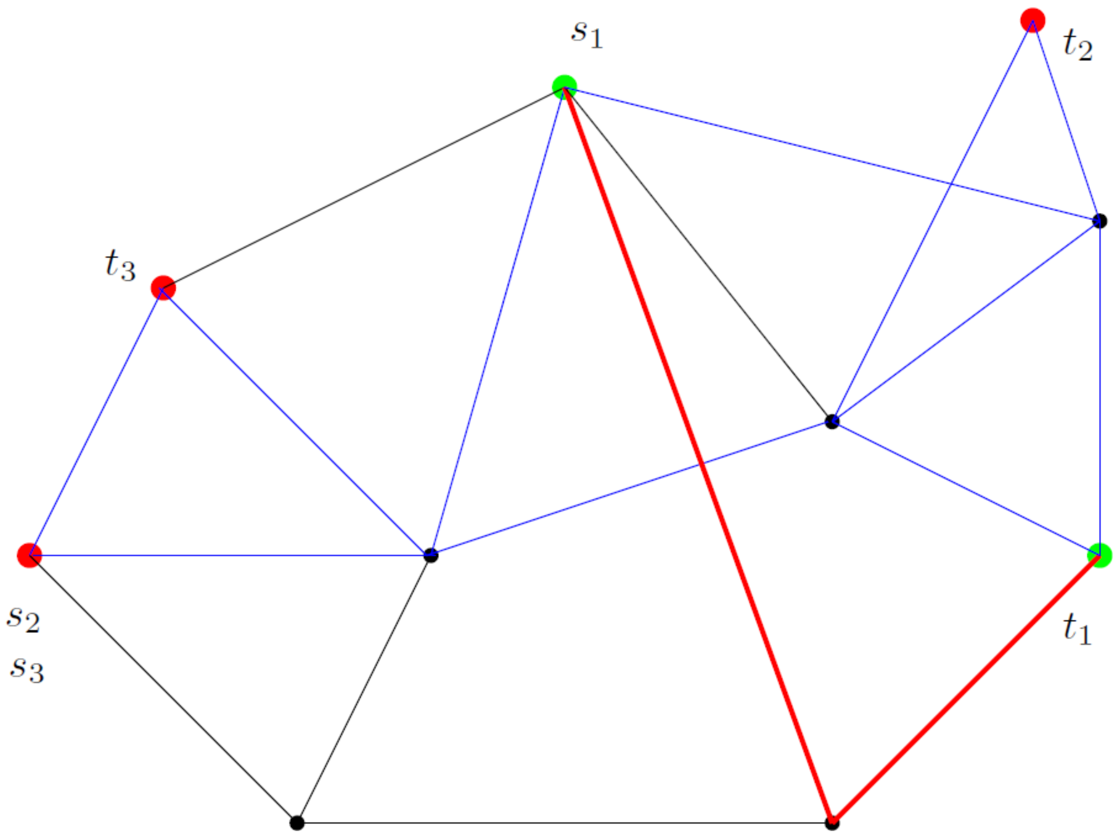
$r(s_1 t_1) = 3$
$r(s_2 t_2) = 2$
$r(s_3 t_3) = 2$

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.



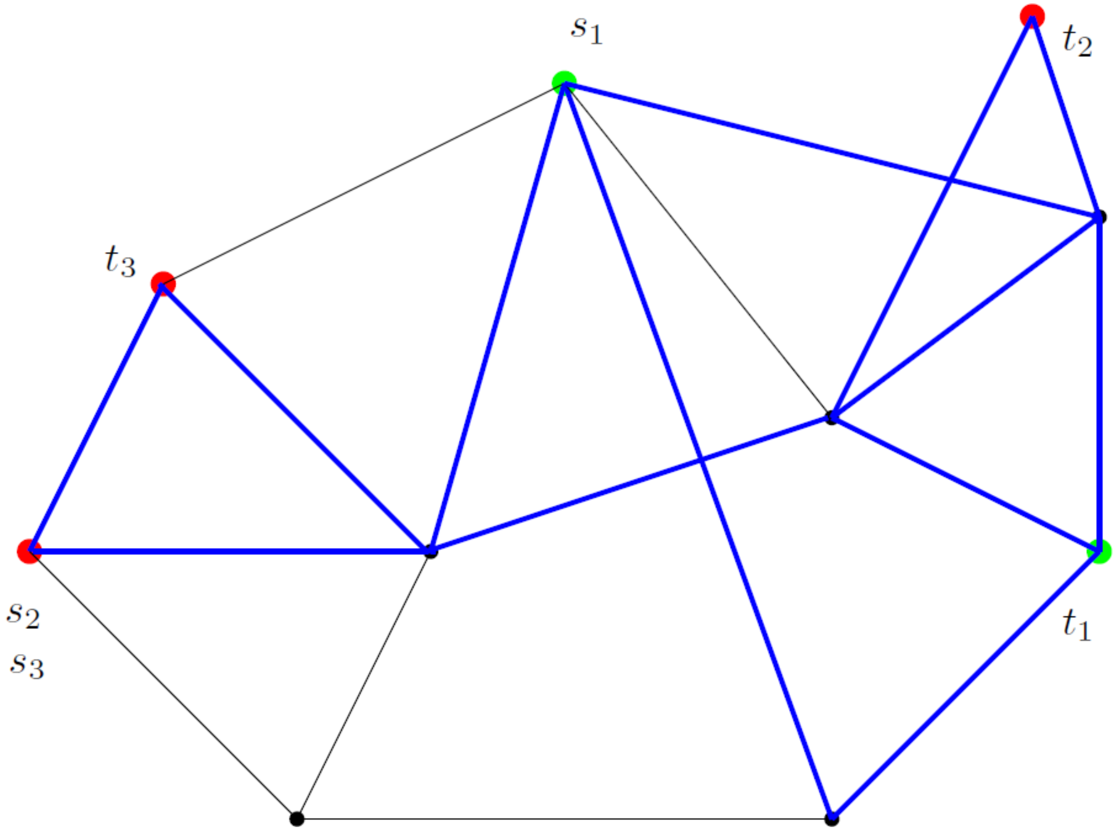
$r(s_1 t_1) = 3$
$r(s_2 t_2) = 2$
$r(s_3 t_3) = 2$

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.



r($s_1 t_1$) = 3
r($s_2 t_2$) = 2
r($s_3 t_3$) = 2

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.
- Nutov gave *combinatorial* $O(\log n)$-approximation solution for each phase.
  - We prove the same ratio by considering Augment-LP relaxation.

Integrality Gap of node-weighted Multiroute-LP

Theorem: The integrality gap of node-weighted Multiroute-LP is $O(k \log n)$.

Idea:

- Use augmentation framework [Williamson et al. '93] & [Nutov '09].
  - In each phase augment the connectivity of unsatisfied pairs by one.
- Nutov gave *combinatorial* $O(\log n)$-approximation solution for each phase.
  - We prove the same ratio by considering Augment-LP relaxation.

In each phase:

$OPT(\text{AugmentLP}) \leq OPT(\text{MultirouteLP})$

Integrality gap of Multiroute-LP is $O(k \log n)$

Integrality Gap of Multiroute-LP

In phase ℓ, increase the connectivity of pairs with requirement at least ℓ and connectivity $\ell - 1$ by at least one

$H_{\ell-1}$: The subgraph selected in the first $\ell - 1$ phases

$G'_\ell = (V, E - E(H_{\ell-1}))$

$h_\ell(S) = 1$ iff $|\delta_{H_{\ell-1}}(S)| = \ell - 1$ and $\max_{r_i \in S, t_i \notin S} r(s_i, t_i) \geq \ell$,
zero otherwise.

Augment-LP(G'_ℓ, h_ℓ)

min $\sum_{v \in V} w(v) x(v)$

s.t. $\sum_{v \in \Gamma_{G'_\ell(S)}} x(v) \geq h_\ell(S) \quad \forall S \subseteq V$

$x(v) \geq 0 \quad \forall v \in V$

Integrality Gap of Multiroute-LP

In phase ℓ, increase the connectivity of pairs with requirement at least ℓ and connectivity $\ell - 1$ by at least one

$H_{\ell-1}$: The subgraph selected in the first $\ell - 1$ phases

$G'_\ell = (V, E - E(H_{\ell-1}))$

$h_\ell(S) = 1$ iff $|\delta_{H_{\ell-1}}(S)| = \ell - 1$ and $\max_{r_i \in S, t_i \notin S} r(s_i, t_i) \geq \ell$, zero otherwise.

Augment-LP(G'_ℓ, h_ℓ)

$$
\begin{aligned}
\min\ & \textstyle\sum_{v \in V} w(v)x(v) & \\
\text{s.t.}\ & \textstyle\sum_{v \in \Gamma_{G'_\ell}(S)} x(v) \geq h_\ell(S) & \forall S \subseteq V \\
& x(v) \geq 0 & \forall v \in V
\end{aligned}
$$

We proved that the integrality gap of Augment-LP is $O(\log n)$ by dual-fitting and spider-cover method

Integrality Gap of Multiroute-LP

In phase ℓ, increase the connectivity of pairs with requirement at least ℓ and connectivity $\ell-1$ by at least one

$H_{\ell-1}$: The subgraph selected in the first $\ell-1$ phases

$G'_\ell = (V, E - E(H_{\ell-1}))$

$h_\ell(S) = 1$ iff $|\delta_{H_{\ell-1}}(S)| = \ell - 1$ and $\max_{r_i \in S, t_i \notin S} r(s_i, t_i) \geq \ell$, zero otherwise.

Augment-LP(G'_ℓ, h_ℓ)

min $\sum_{v \in V} w(v)x(v)$

s.t. $\sum_{v \in \Gamma_{G'_\ell}(S)} x(v) \geq h_\ell(S) \quad \forall S \subseteq V$

$x(v) \geq 0 \quad \forall v \in V$

Dual of Augment-LP(G'_ℓ, h_ℓ)

max $\sum_{S \subseteq V} h_\ell(S) y(S)$

s.t. $\sum_{S: v \in \Gamma_{G'_\ell}(S)} y(S) \leq w(v) \quad \forall v \in V$

$y(S) \geq 0 \quad \forall S \subseteq V$

Integrality Gap of Multiroute-LP

In phase ℓ, increase the connectivity of pairs with requirement at least ℓ and connectivity $\ell - 1$ by at least one

$H_{\ell-1}$: The subgraph selected in the first $\ell - 1$ phases

$G'_\ell = (V, E - E(H_{\ell-1}))$

$h_\ell(S) = 1$ iff $|\delta_{H_{\ell-1}}(S)| = \ell - 1$ and $\max_{r_i \in S, t_i \notin S} r(s_i, t_i) \geq \ell$, zero otherwise.

Augment-LP(G'_ℓ, h_ℓ)

min $\sum_{v \in V} w(v) x(v)$

s.t. $\sum_{v \in \Gamma_{G'_\ell}(S)} x(v) \geq h_\ell(S) \quad \forall S \subseteq V$

$x(v) \geq 0 \quad \forall v \in V$

We proved that the integrality gap of Augment-LP is $O(\log n)$ by dual-fitting and spider-cover method

In contrary to the edge-weighted case, integrality gap is *unbounded* for an arbitrary uncrossable function h. It just holds for the functions arise from an SNDP instance.

Planar Graphs

Integrality gap of Augment-LP in planar graph is $O(1)$ [Chekuri et al. '12]

$O(k)$-approximation for node-weighted SNDP on planar graphs

$O(k^2)$-approximation for node-weighted PC-SNDP on planar graphs

Questions?

Thanks!

Different LP Relaxation

It is *not* based on Multiroute flow!
In a feasible solution H, s and t are r(s,t)-connected.

[Menger's theorem] By omitting $\ell < r(s,t)$ edges form H, s and t remain connected.

We can write an LP-relaxation for SNDP problem based on this property.

The exact optimal solution can be found.

However; its running time is polynomial in n^k

K-approximate Solution to Multiroute-LP

Compact-PC-Multiroute-LP

min $\sum_{v \in V} w(v)x(v) \sum_{st \in V \times V} \pi(st)z(st)$

s.t. $\sum_{a \in \delta^+(s)} f(a,st) - \sum_{a \in \delta^-(s)} f(a,st) \geq \big(1 - z(st)\big)r(st) \quad \forall st$

$\sum_{a \in \delta^+(v)} f(a,st) = \sum_{a \in \delta^-(s)} f(a,st) \quad \forall st, \forall v \notin \{s,t\}$

$f(a,st) \leq 1 - z(st) \quad \forall a, \forall st$

$\sum_{a \in \delta^-(v)} f(a,st) \leq r(st)x(v) \quad \forall st, \forall v$

$0 \leq z(st) \leq 1 \quad \forall st$

$0 \leq x(v) \leq 1 \quad \forall v$

$f(a,st) \geq 0 \quad \forall a, \forall st$

K-approximate Solution to Multiroute-LP

Compact-PC-Multiroute-LP

min $\sum_{v \in V} w(v)x(v) \sum_{st \in V \times V} \pi(st)z(st)$

s.t. $\sum_{a \in \delta^+(s)} f(a,st) - \sum_{a \in \delta^-(s)} f(a,st) \geq \big(1 - z(st)\big) r(st) \quad \forall st$

$\sum_{a \in \delta^+(v)} f(a,st) = \sum_{a \in \delta^-(s)} f(a,st) \quad \forall st, \forall v \notin \{s,t\}$

$f(a,st) \leq 1 - z(st) \quad \forall a, \forall st$

$\sum_{a \in \delta^-(v)} f(a,st) \leq r(st)x(v) \quad \forall st, \forall v$

$0 \leq z(st) \leq 1 \quad \forall st$

$0 \leq x(v) \leq 1 \quad \forall v$

$f(a,st) \geq 0 \quad \forall a, \forall st$

Decomposition Lemma

[Aggarwal and Orlin '02][Kishimoto '96]

Thm:
$G = (V, A)$ be a directed graph.
s and t be two vertices of V
f: an s-t flow of value $k\rho$ (integer k and real value ρ) such that $f(a) \leq \rho$.

There exists a **k-route** flow of value **r** that preserves flow of each edge