

Improved Approximation Algorithms for Degree-bounded Network Design Problems with Node Connectivity Requirements

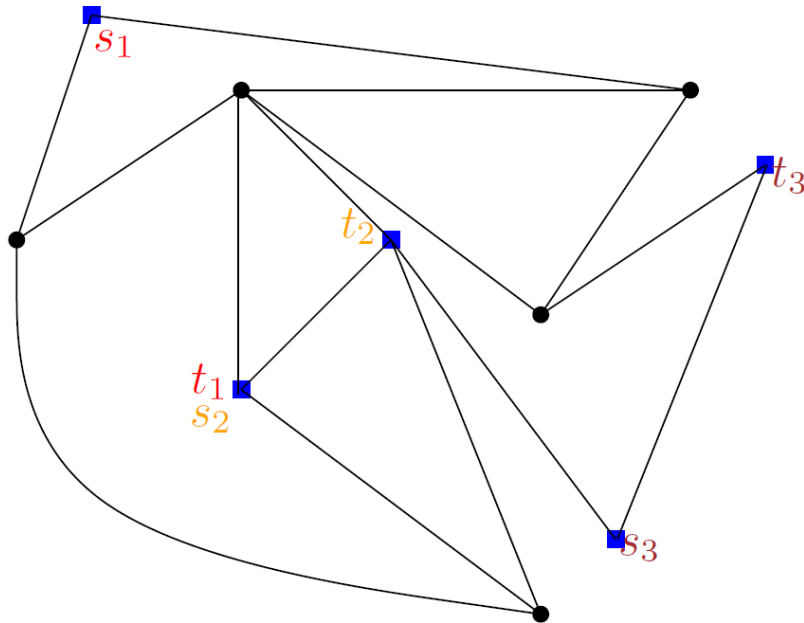
Alina Ene



Ali Vakilian



Survivable Network Design (SNDP)



Given an **edge weighted** graph G

Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

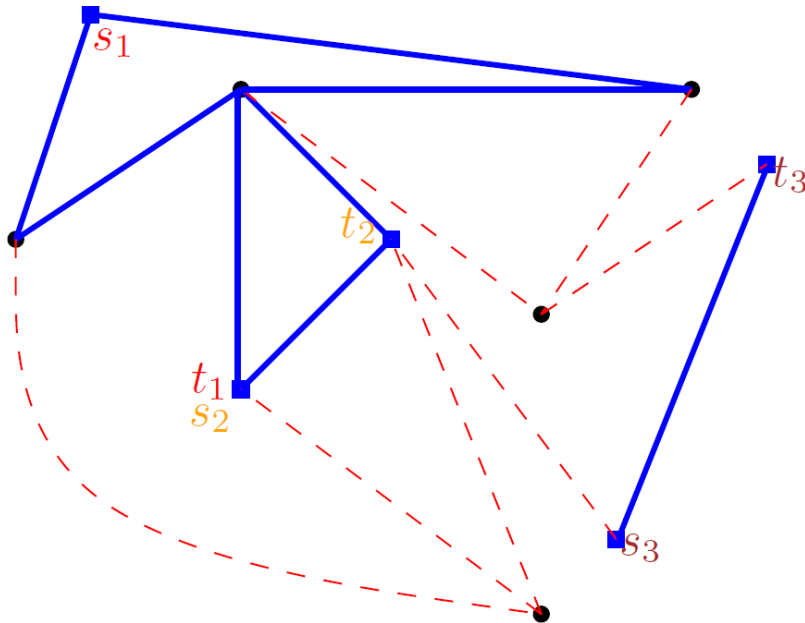
Goal: Min weight subgraph H
containing $r(s_i, t_i)$ **disjoint** paths
between s_i and t_i .

$$r(s_1, t_1) = 2$$

$$r(s_2, t_2) = 2$$

$$r(s_3, t_3) = 1$$

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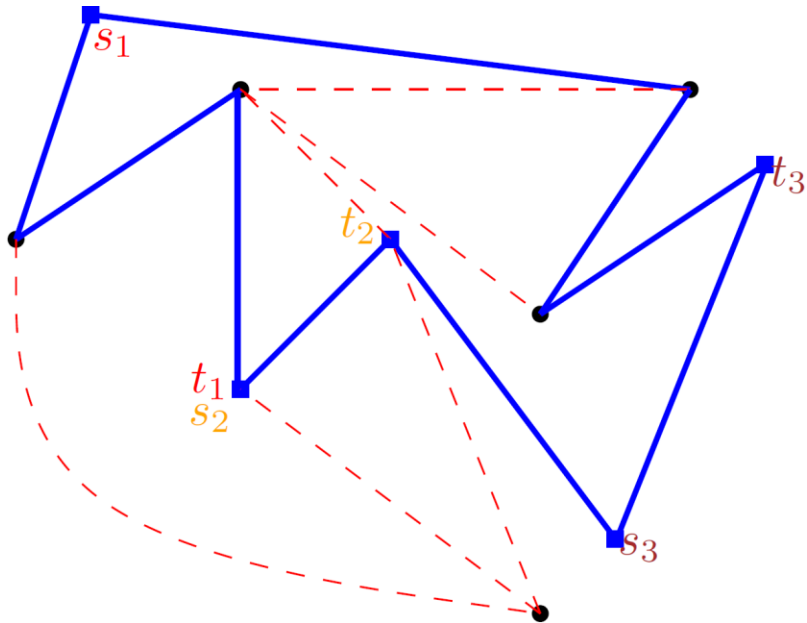
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Edge Connectivity

max number of **edge-disjoint** paths

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Goal: Min weight subgraph H
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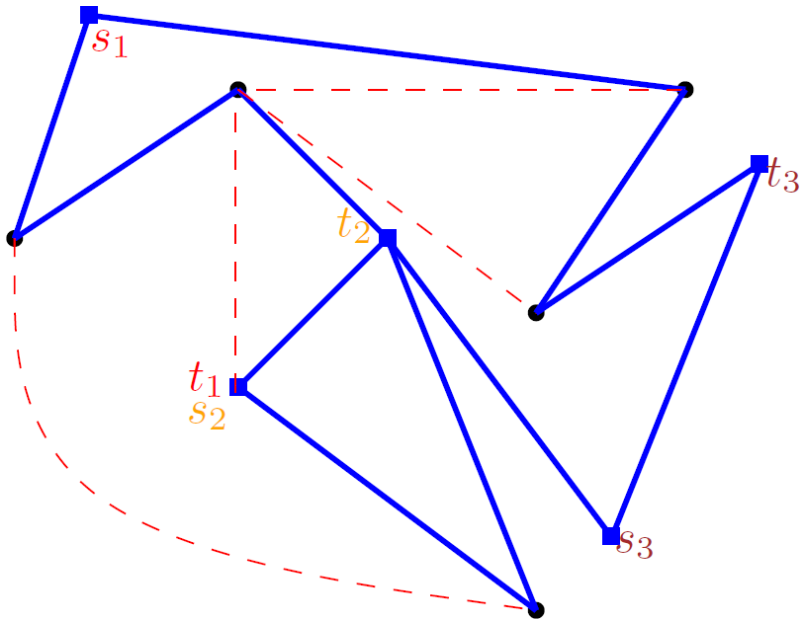
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Node Connectivity

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Goal: Min weight subgraph H
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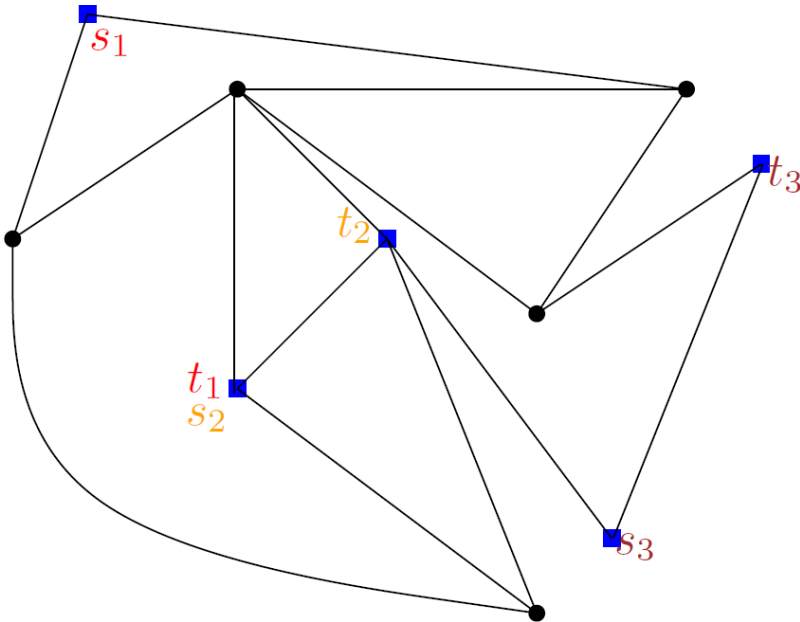
Element Connectivity
max number of **element-disjoint** paths

Survivable Network Design (SNDP)

k = maximum requirement; $\max_{i \leq l} r(s_i, t_i)$

Edge conn.	Element conn.	Node conn.
2-approx [Jain '98]	2-approx [Fleischer et al. '01]	$O(k^3 \log n)$ [Chuzhoy-Khanna '09]

Degree bounded SNDP



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Collection of l pairs: $(s_1, t_1), \dots, (s_l, t_l)$

$r(s_i, t_i)$: requirement of pair (s_i, t_i)

$b(v)$: degree bound of node v

Goal: Min weight subgraph H containing $r(s_i, t_i)$ **edge-disjoint** paths between s_i and t_i and degree of each node v is at most $b(v)$.

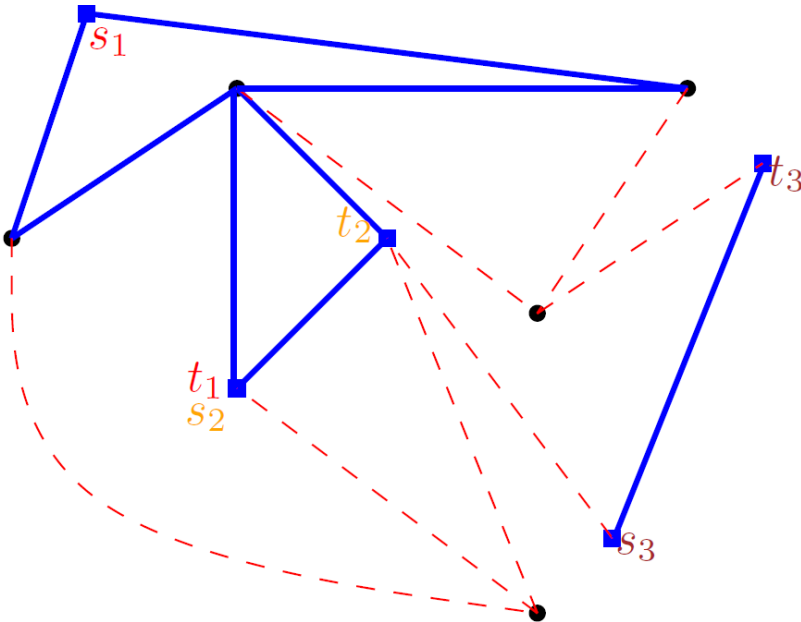
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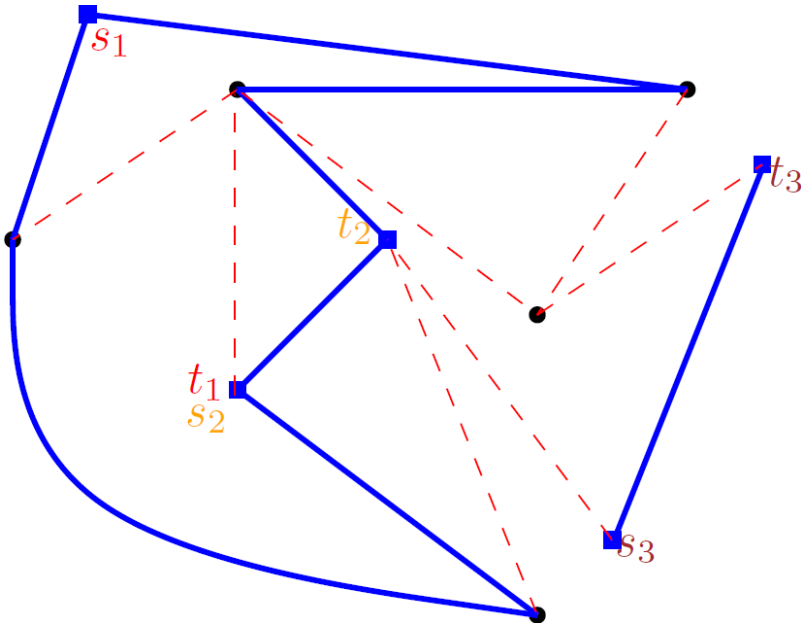
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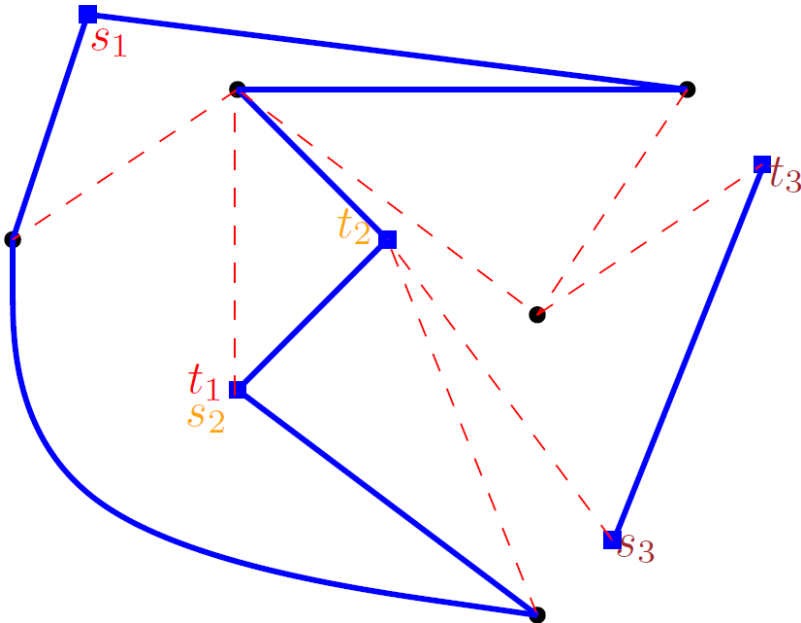
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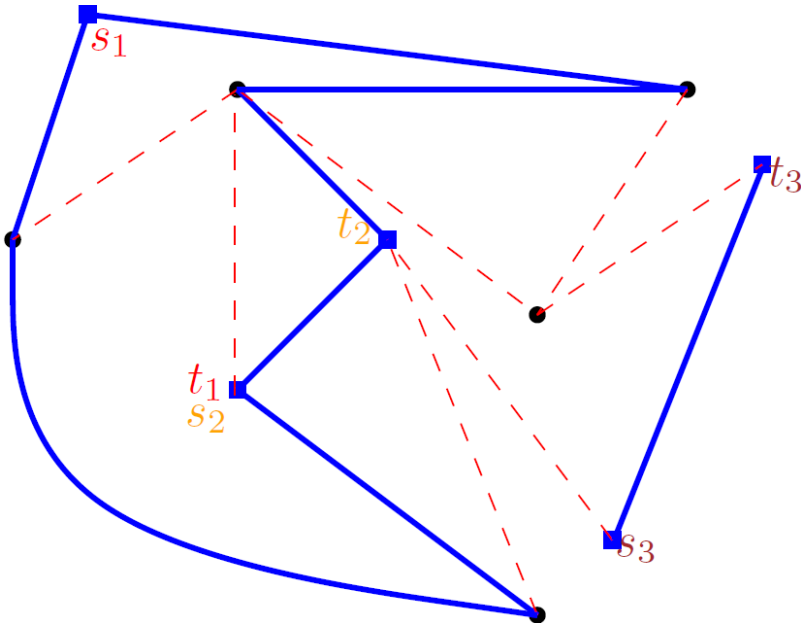
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Bad News:

By a reduction to the **Hamiltonian Path** problem it is even NP-hard to find a feasible solution of **Deg-SNDP**.

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Bad News:

By a reduction from the **3-Partition** problem it is even NP-hard to approximate the cost of **Deg-SNDP**.

Bicriteria
Approximation
(cost & degree)

h problem it is
ation of **Deg-SNDP**.

Bicriteria Approximation

A subgraph $H = (V, E_H)$ is an $(\alpha, \beta b(v) + \gamma)$ -approximation for Deg-SNDP(G, c, b) problem iff

$$c(E_H) \leq \alpha \cdot c(\text{OPT})$$

and $d_H(v) \leq \beta(b(v)) + \gamma$ for each $v \in V$

Degree Bounded SNDP

k = maximum requirement; $\max_{i \leq l} r(s_i, t_i)$

Edge conn.	Element conn.	Node conn.
$(2, 2b(v)+3)$ [Lau et al '07]	$(O(\log k), O(2^k)b(v))$ [Nutov '12]	$O(k^3 \log n) \times (O(\log k), O(2^k)b(v))$ [Nutov '12, Chuzhoy and Khanna '09]
$(2, b(v)+O(k))$ [Lau and Singh '08]	$(O(k), O(k)b(v))$ [Fukunaga and Ravi '12]	$O(k^3 \log n) \times (O(k), O(k)b(v))$ [Fukunaga and Ravi '12 Chuzhoy and Khanna '09]
$(2, 2b(v)+2)$ [Louis and Vishnoi '10]	$(O(1), O(1)b(v) + O(k))$ [Fukunaga, Nutov and Ravi '13]	$O(k^3 \log n) \times (O(1), O(1)b(v) + O(k))$ [Fukunaga, Nutov and Ravi '13 Chuzhoy and Khanna '09]
2-approx. $\min\{2b(v)+2, b(v)+3k\}$ [Lau and Zhou '14]	$(O(1), O(1)b(v))$ [Ene and V. '14]	$O(k^3 \log n) \times (O(1), O(1)b(v))$ [Ene and V. '14, Chuzhoy and Khanna '09]

- ❑ Iterative rounding
- ❑ Degree-bounded Edge Connectivity SNDP
- ❑ Challenges with Node-Connectivity Req

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If x is an extreme point solution of EC-SNDP-LP, then one of the following holds [Jain '98]

There is an edge e such that $x(e) = 0$

There is an edge e such that $x(e) > \frac{1}{2}$

EC-SNDP-LP(E, r)

$$\min \sum_{e \in E} c(e)x(e)$$

$$\text{s.t. } \sum_{e \in \delta(S)} x(e) \geq r(S) \quad \forall S \subseteq V$$

$$0 \leq x(e) \leq 1 \quad \forall e \in E$$

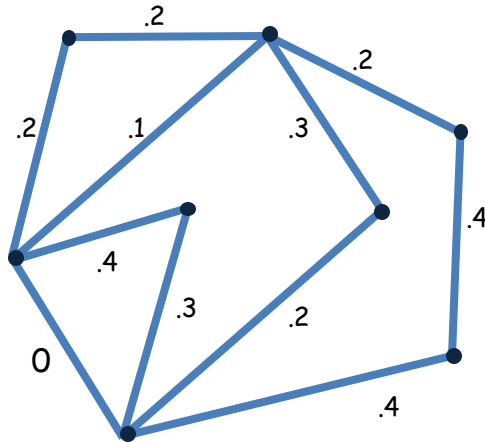
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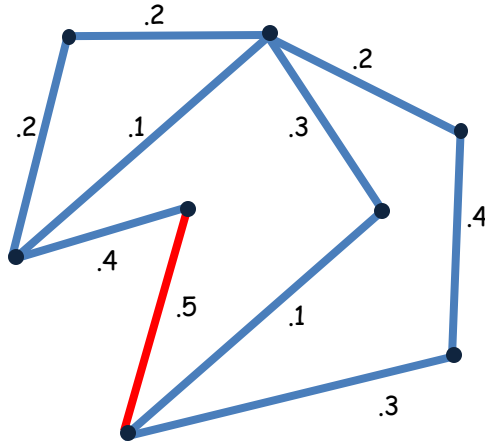
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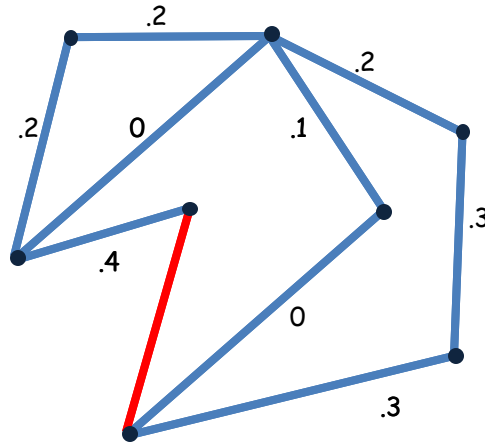
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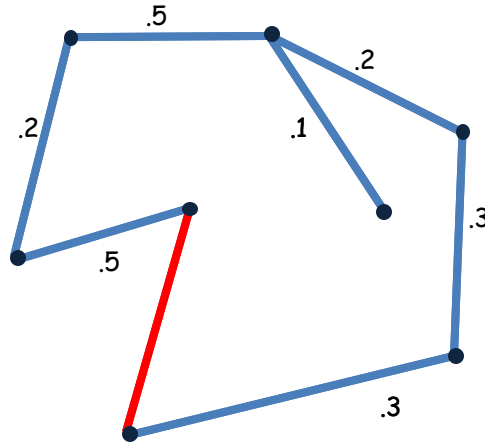
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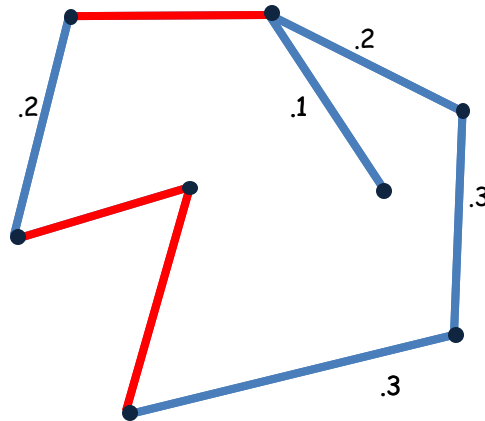
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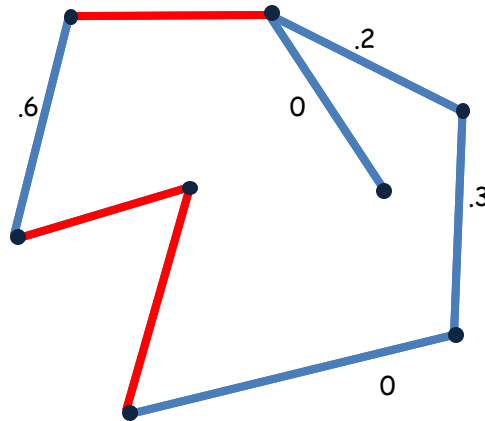
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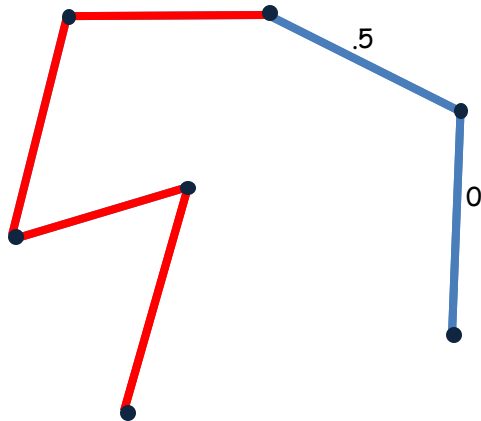
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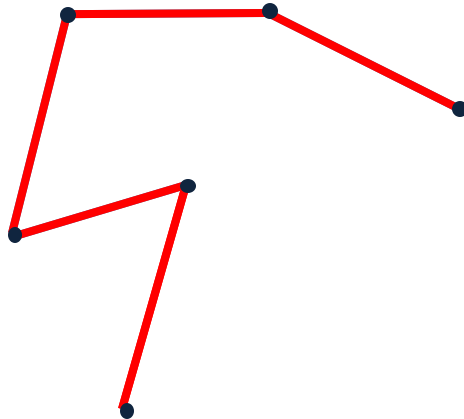
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 - 2-approximation for EC-SNDP [Jain '98]
- ❑ Works for Deg-bounded variants with edge connectivity
 - $(1, b(v)+1)$ -approximation for Deg-MST [Singh and Lau '07]
 - $(2, O(1)b(v))$ -approximation for Deg-EC-SNDP [Lau et al '07]
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It gets more difficult when we consider degree-bound constraints with **node-connectivity** requirements!

How to analyze?

Degree bounded Edge
Connectivity SNDP

Edge Connectivity SNDP

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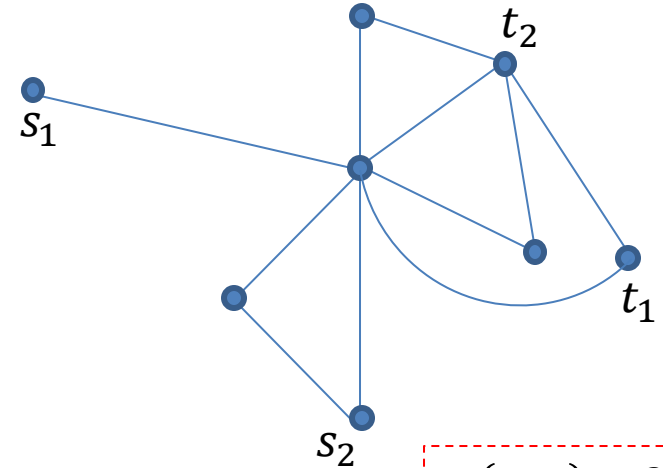
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$$\begin{aligned} r(s_1 t_1) &= 2 \\ r(s_2 t_2) &= 1 \end{aligned}$$

Menger theorem: s and t are k edge-connected \iff st min-cut $\geq k$

$$\text{Let } r(S) = \max_{s_i \in S, t_i \in V \setminus S} r(s_i t_i)$$

Edge set F is a feasible solution

$$\iff \delta_F(S) \geq r(S) \text{ for each } S$$

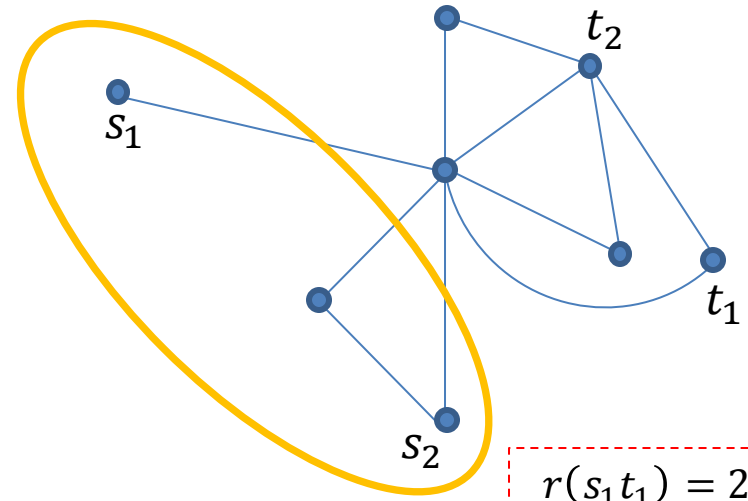
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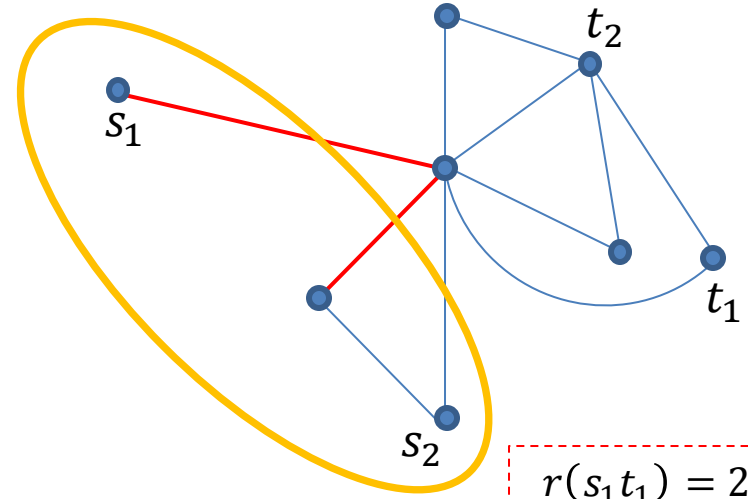
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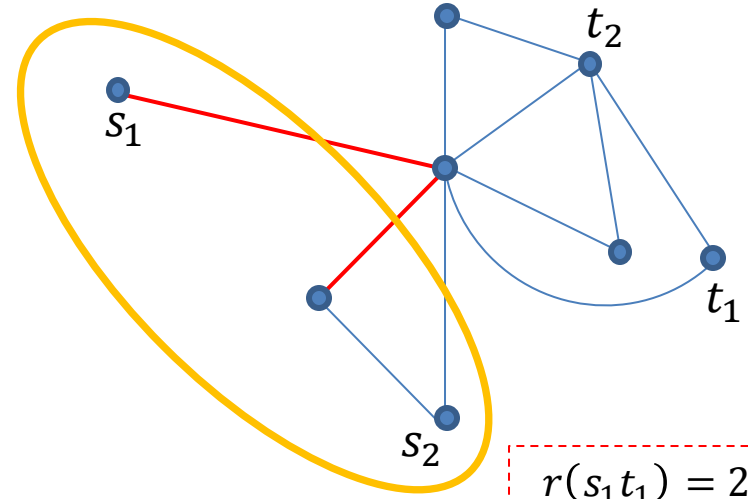
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$$\sum_{e \in \delta(v)} x(e) \leq b(v) \quad \forall v \in V$$

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Main Theorem

Theorem. If x is an extreme point solution of **Deg-EC-SNDP-LP**, then one of the following holds

There is an edge e such that $x(e) = 0$

There is an edge e such that $x(e) > \frac{1}{3}$

There is a vertex v such that $|\delta_E(v)| \leq 6$

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Theorem. If x is an extreme point solution of Deg-EC-SNDP-LP, then one of the following holds

There is an edge e such that $x(e) = 0$

There is an edge e such that $x(e) > \frac{1}{3}$

There is a vertex v such that $|\delta_E(v)| \leq 6$

$(3, 3b(v) + 6)$ -approximation for Deg-EC-SNDP

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Proof. By contradiction, assume that

For each e , $0 < x(e) < \frac{1}{3}$

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Proof of Main Theorem

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For extreme point solution x ,

- A set of tight sets L
- A set of tight vertices C

Deg-EC-SNDP-LP(E, r, b)

$$\min \sum_{e \in E} c(e)x(e)$$

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S is a **tight set** iff

$$\sum_{e \in \delta(S)} x(e) = r(S) > 0$$

v is a **tight vertex** iff

$$\sum_{e \in \delta(v)} x(e) = b(v) > 0$$

Proof of Main Theorem

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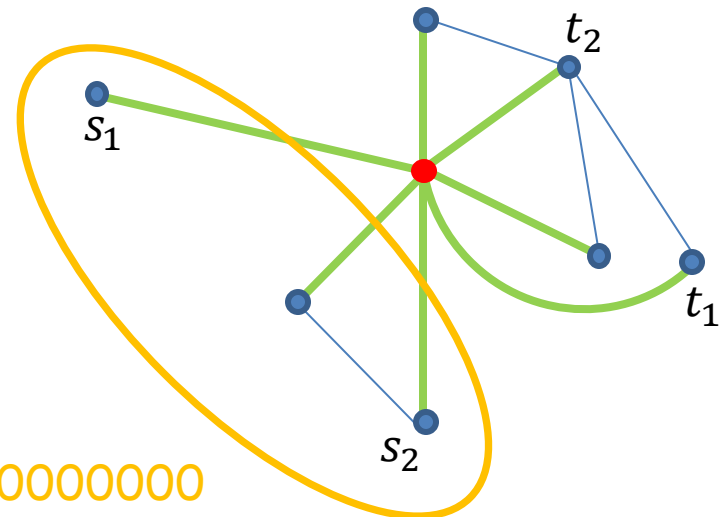
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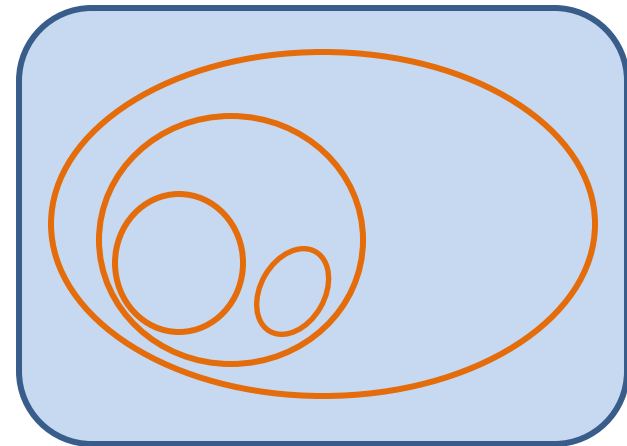
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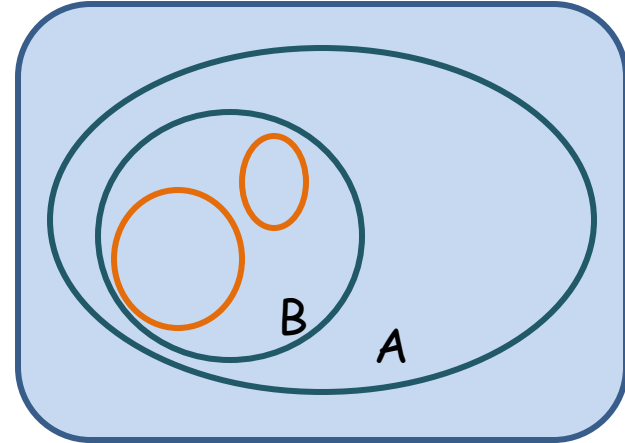
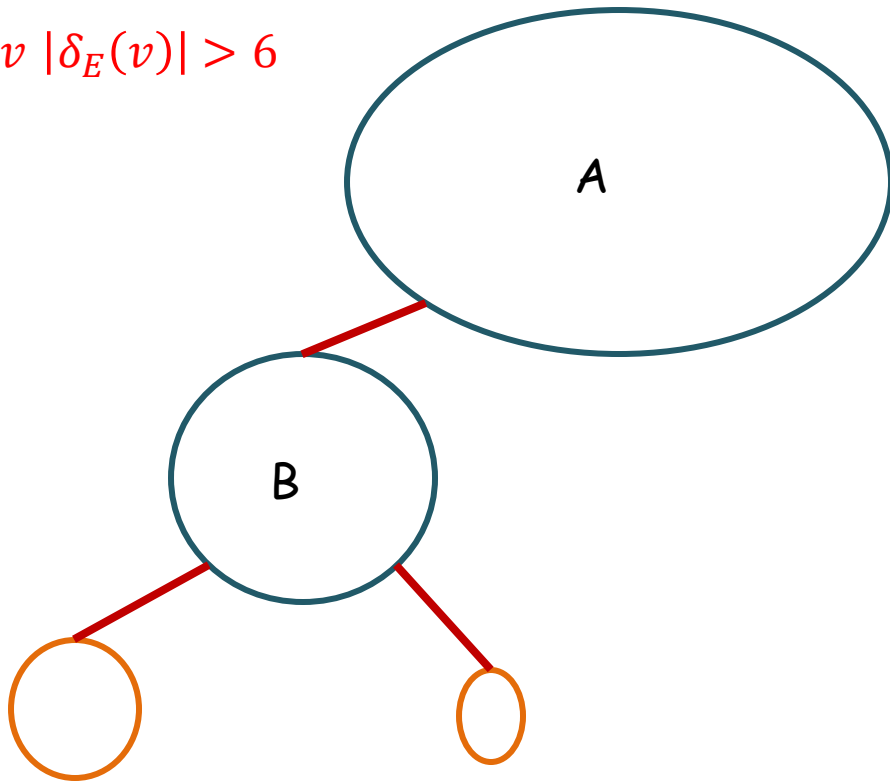
- $|L| + |C| = |E|$
- L and C are linearly independent
- L is a laminar family [Jain '98]



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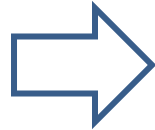


Tree (forest) representation of a laminar family

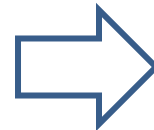
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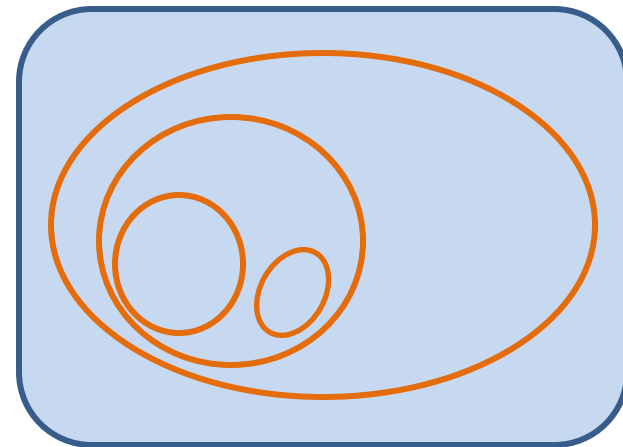
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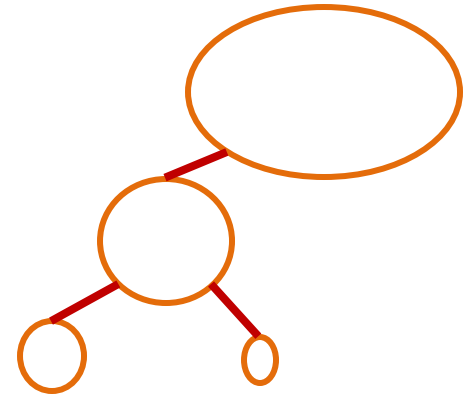
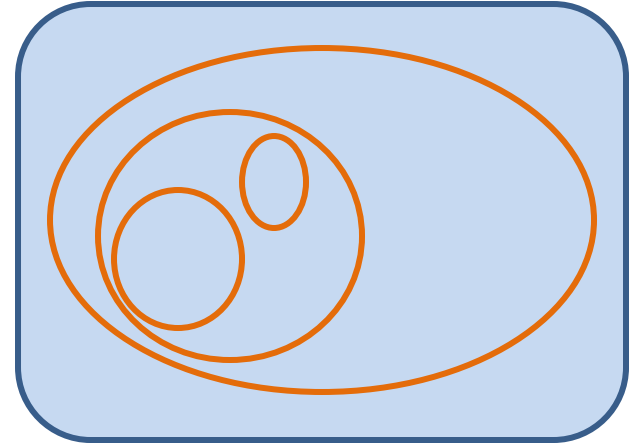
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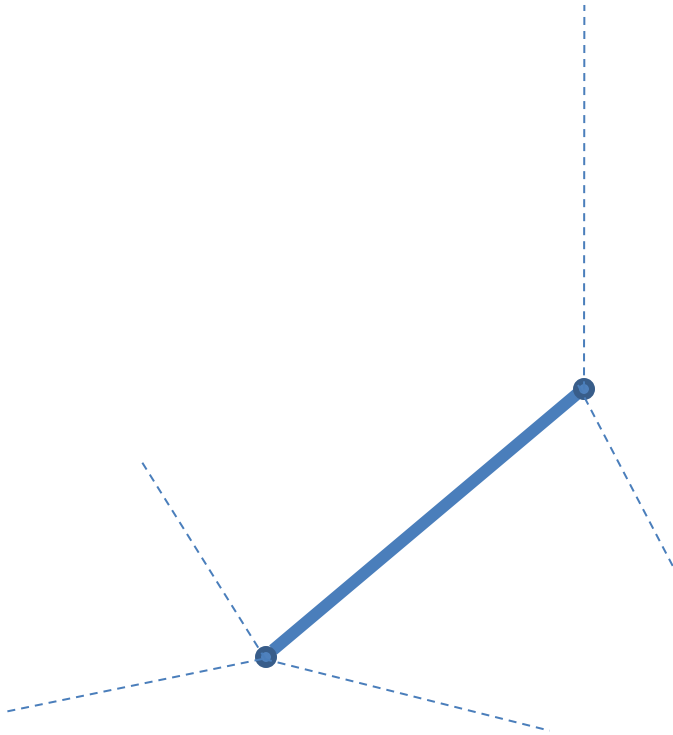
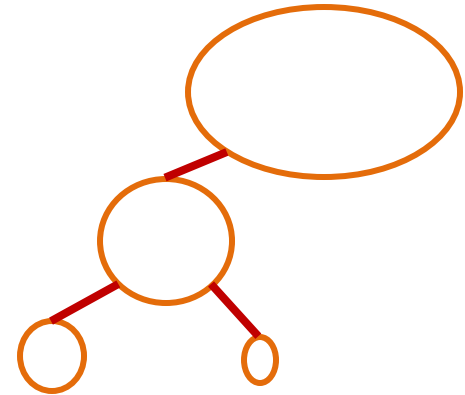
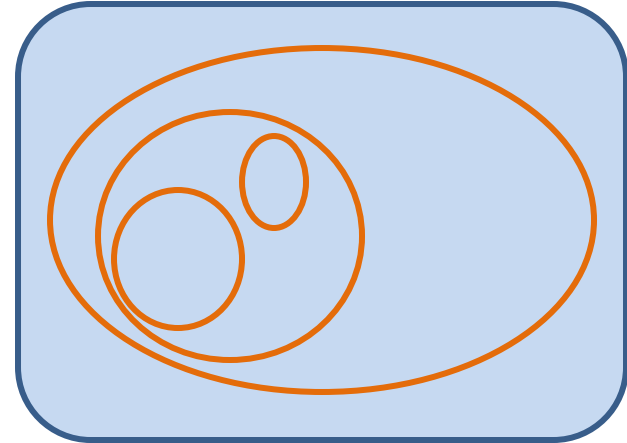
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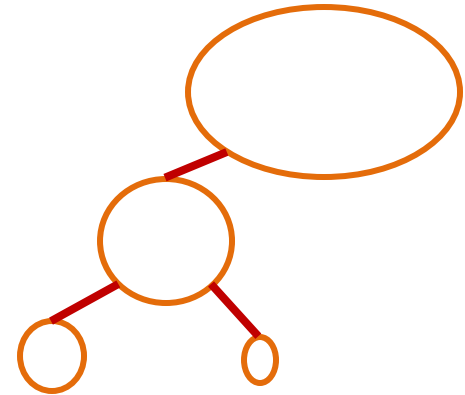
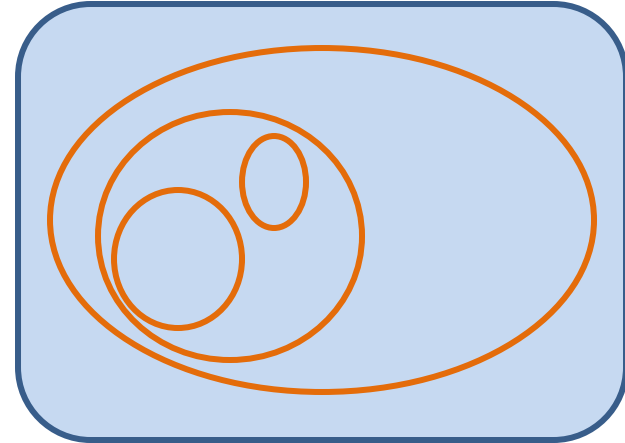
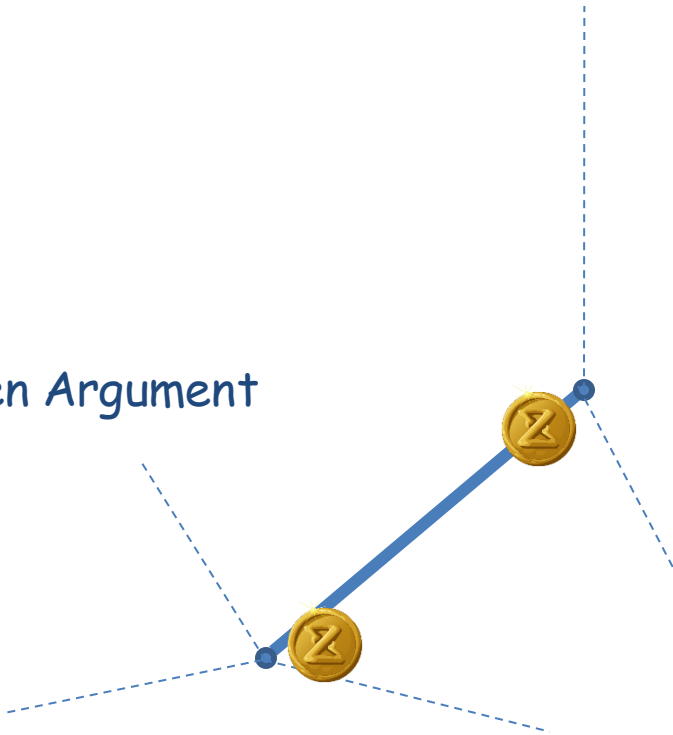


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Token Argument



Proof of Main Theorem

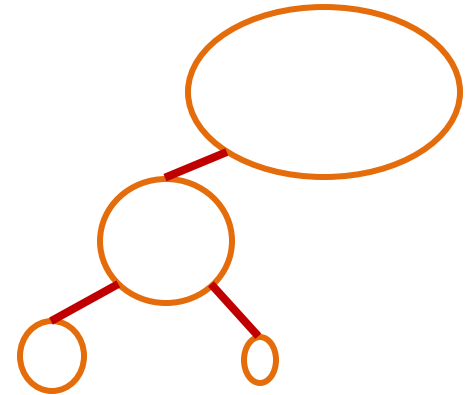
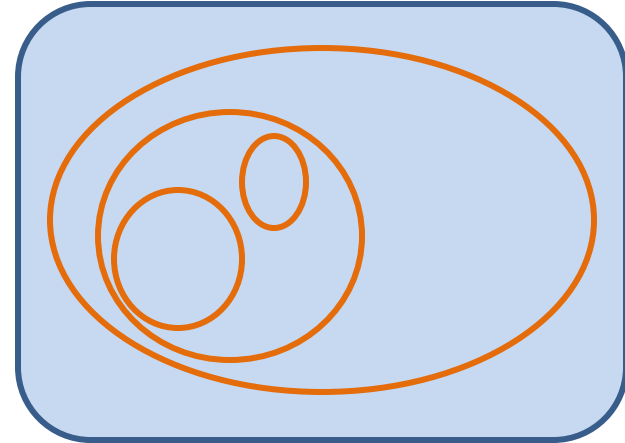
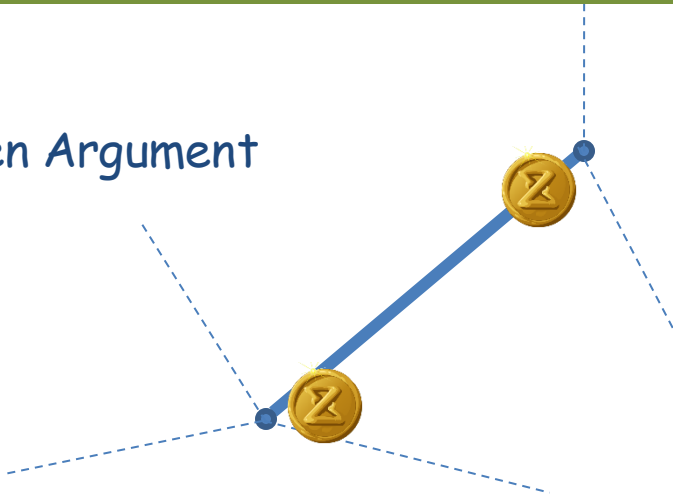
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Idea:

Rearrange tokens such that each node in the "tree rep." and each vertex in C receives two tokens and root vertices receives four tokens.

Token Argument



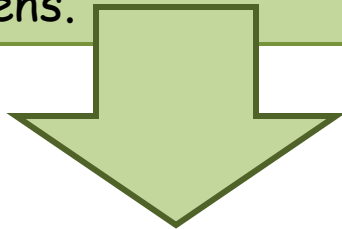
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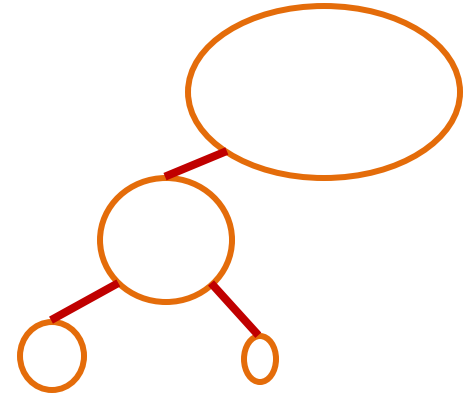
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$$\# \text{of tokens} = 2|E| > 2(|L| + |C|)$$

Contradiction!



For solution x , there exist

- A set of tight sets L
- A set of vertices sets C

Such that

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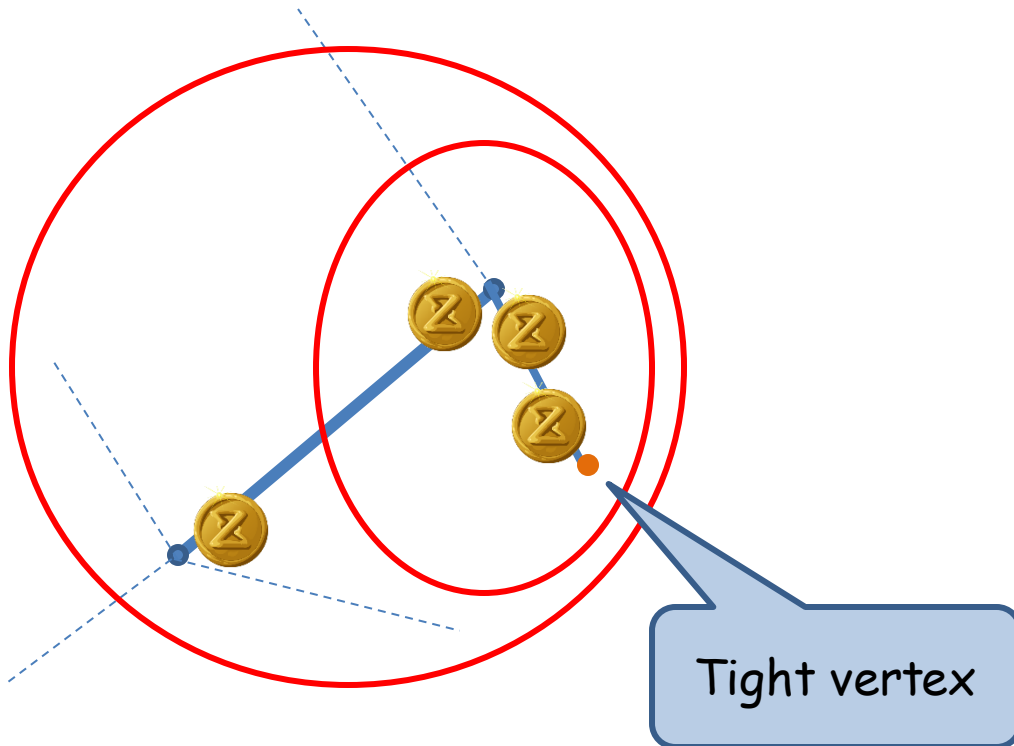
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Initial token assignment

Consider an edge $e=uv$. The token corresponding to $e_v, t_{e,v}$ is assigned to the minimal set in $L \cup C$ that contains v .



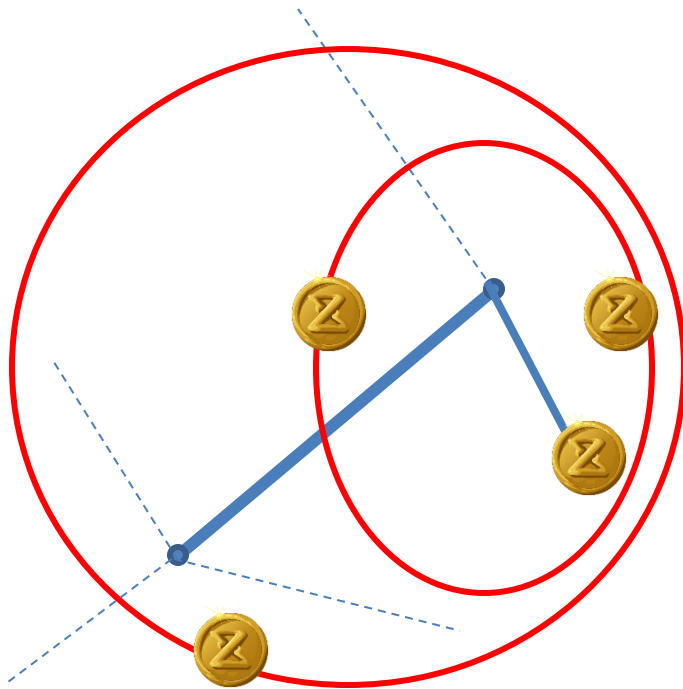
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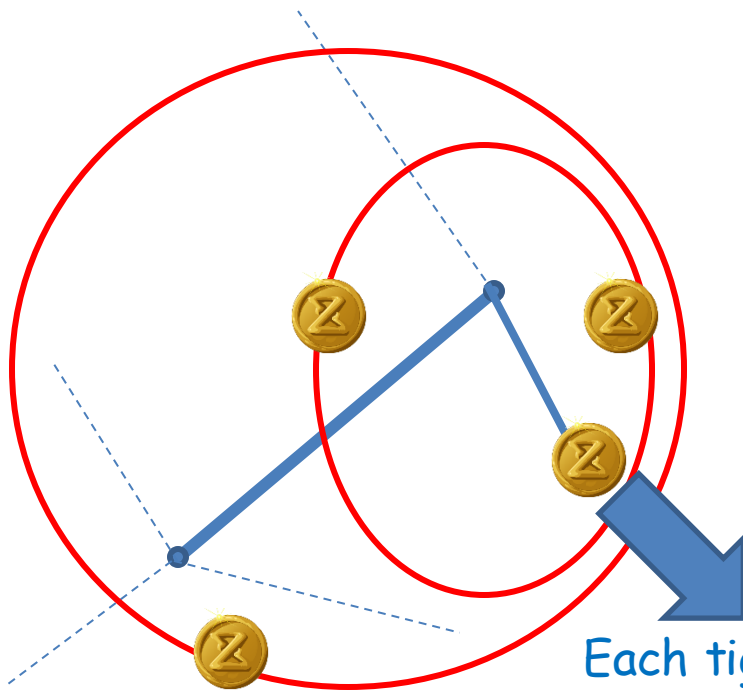
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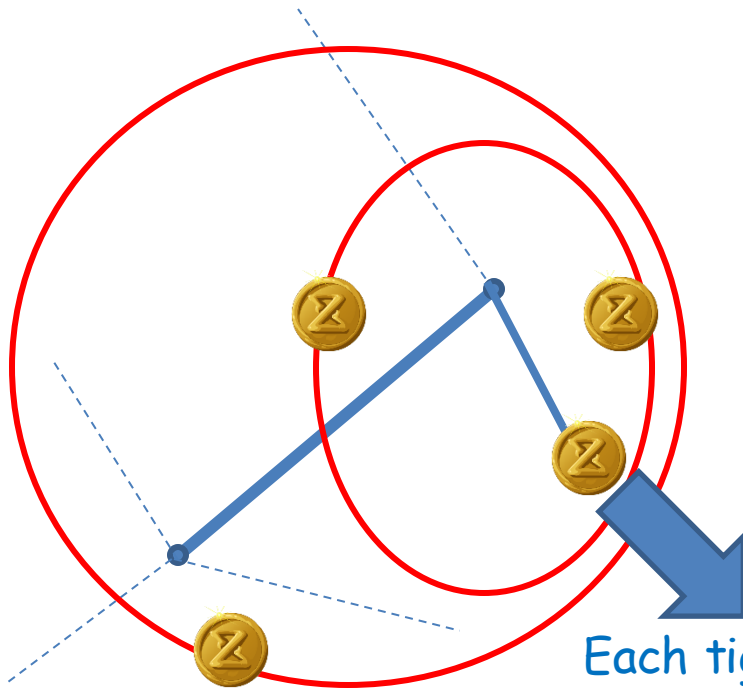
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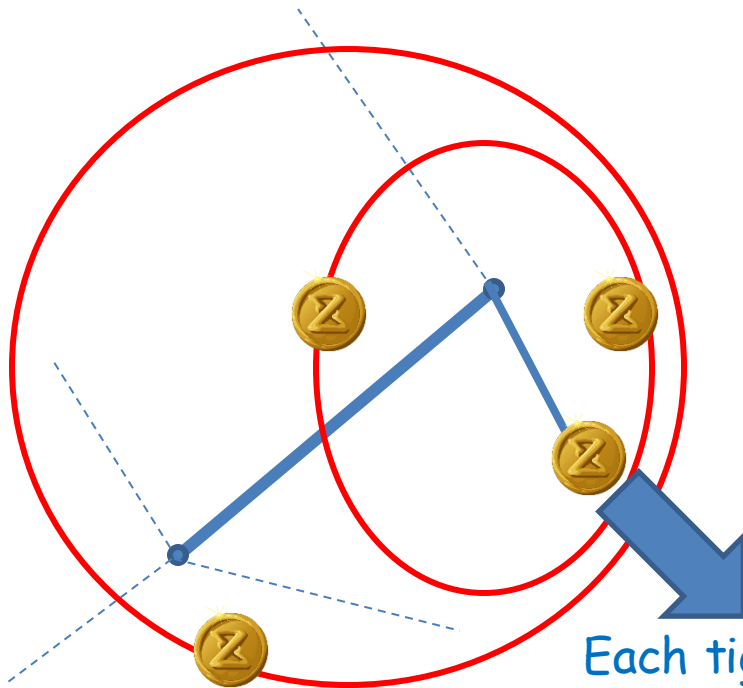
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Each tight vertex receives at least two tokens

- Redistribute along the "tree"

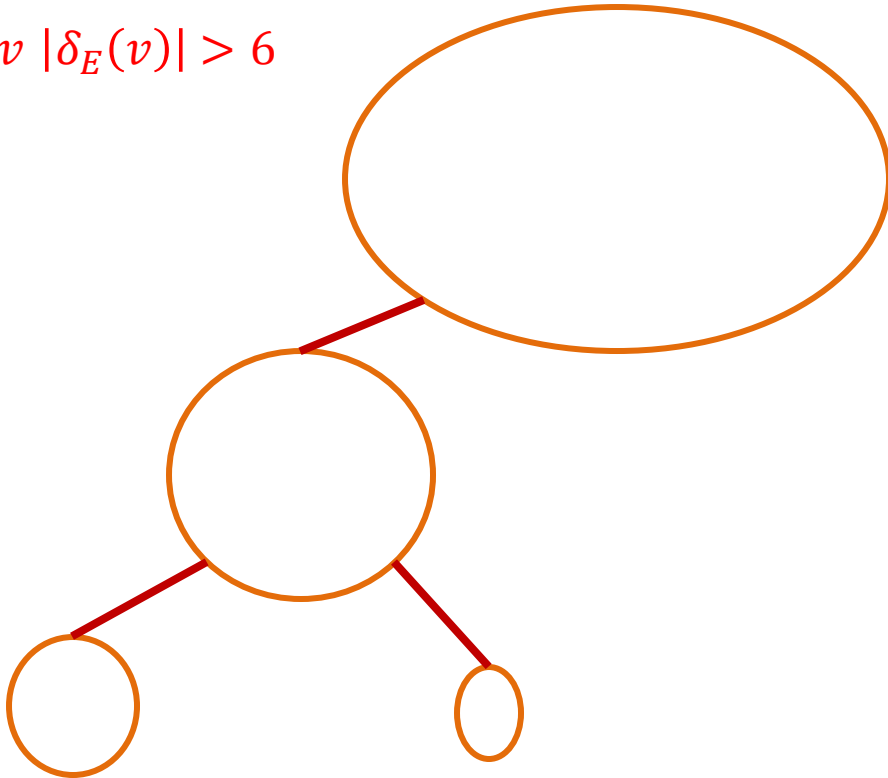


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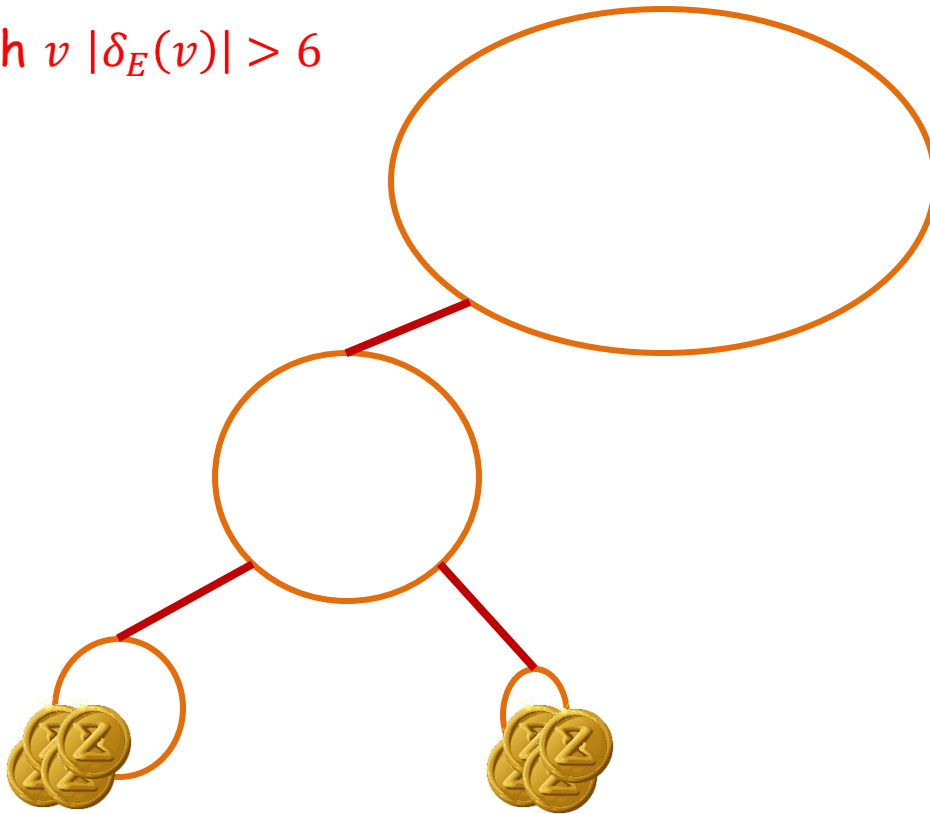


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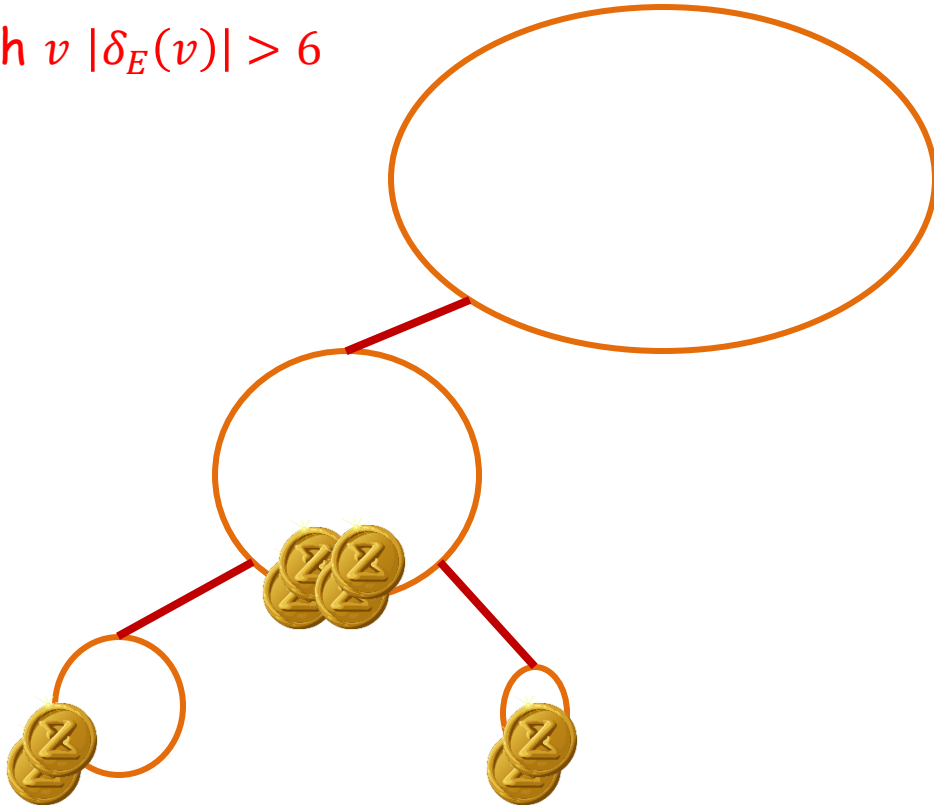
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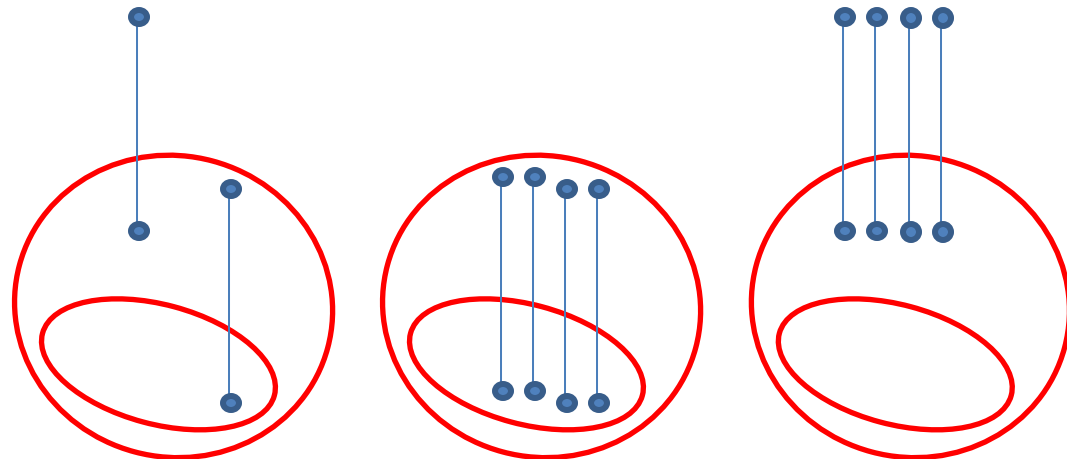
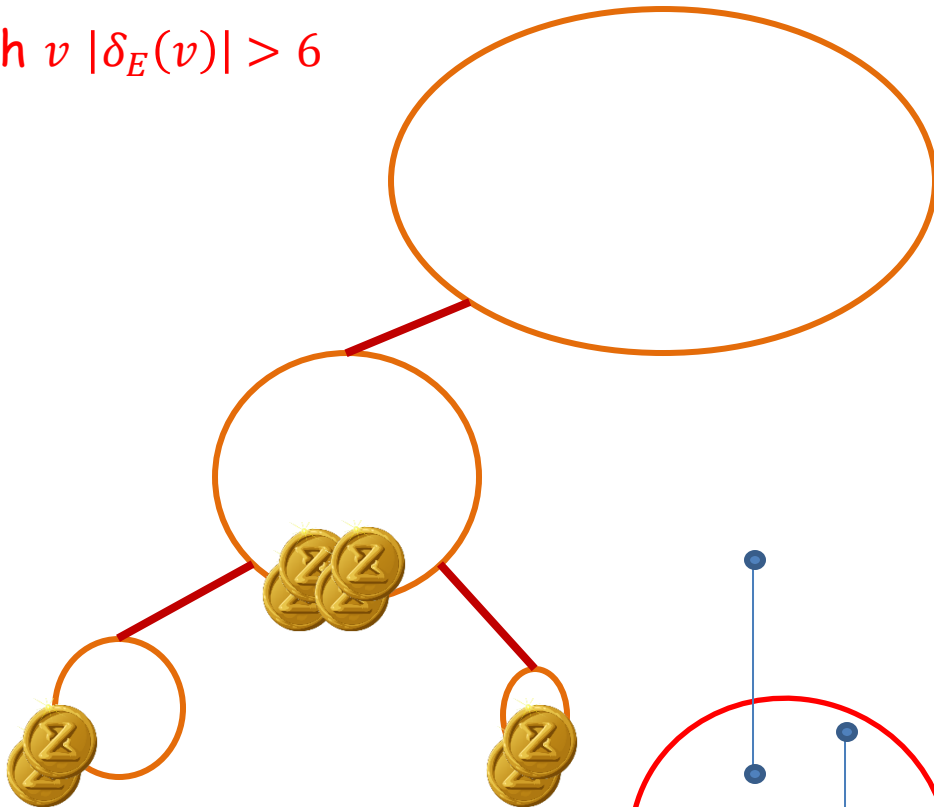


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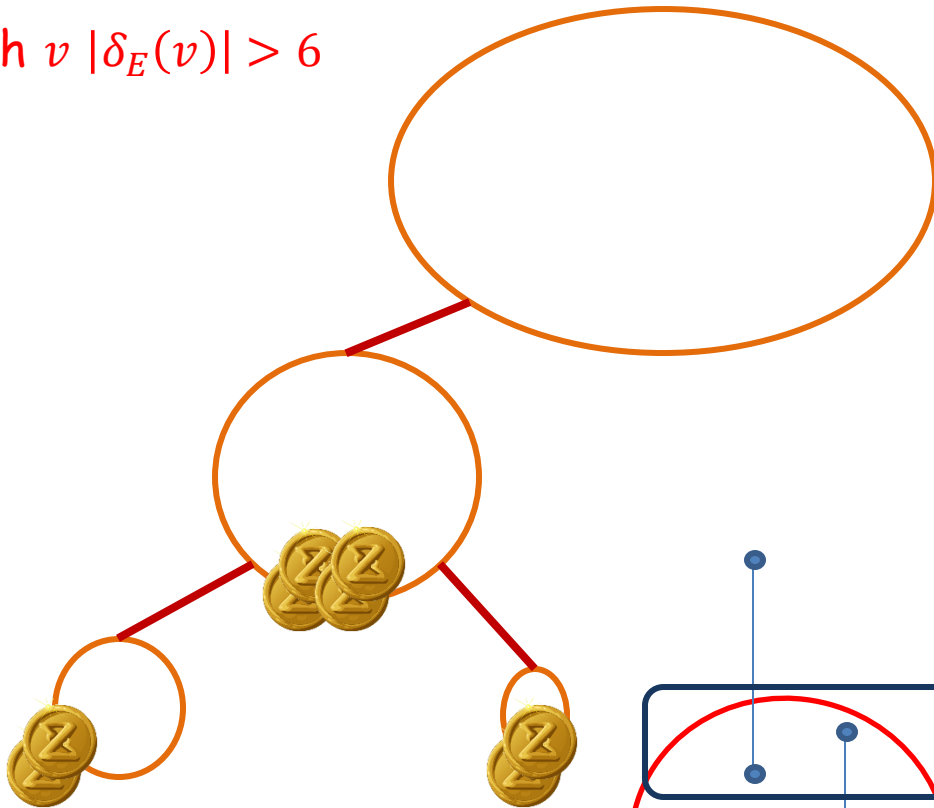


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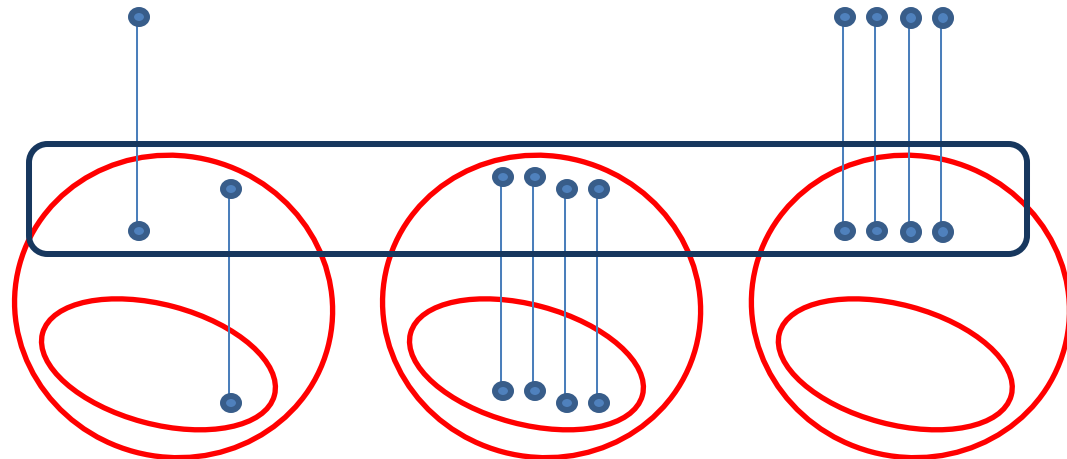
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Back to Node Connectivity!

Element Connectivity

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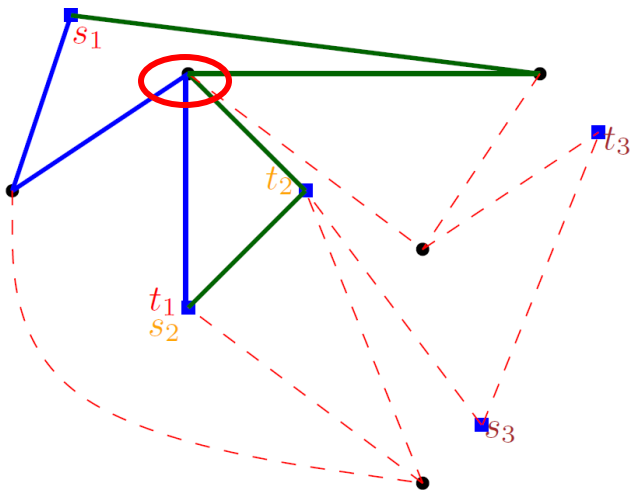
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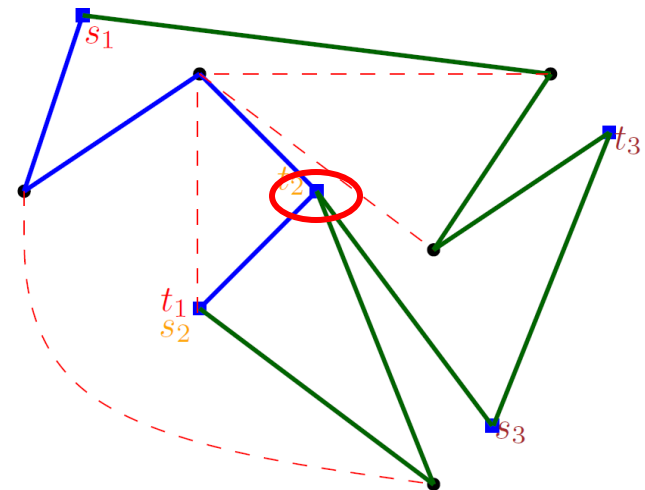
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$s_1 t_1$ -paths share non-terminal



$s_1 t_1$ -paths share terminal

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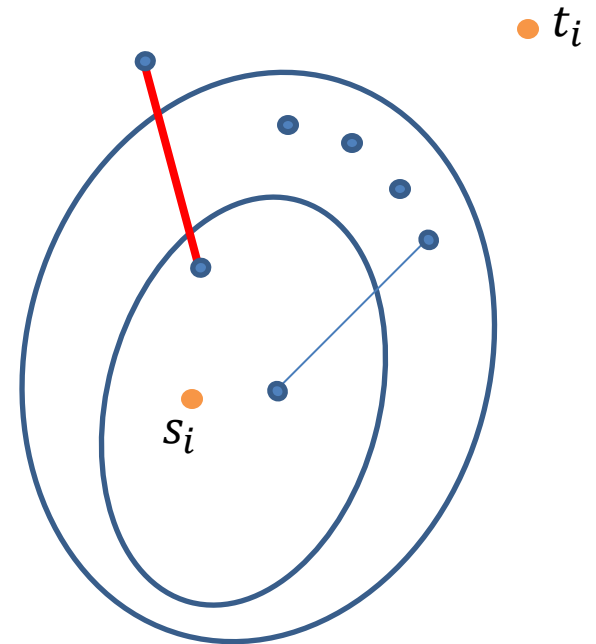
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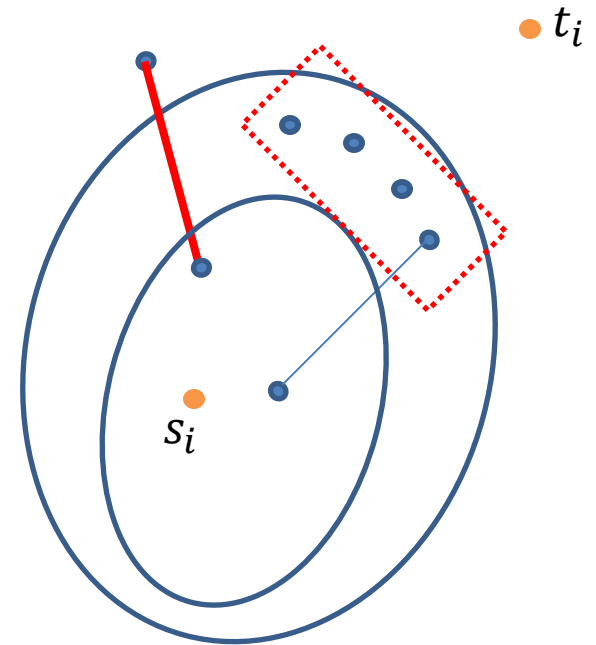
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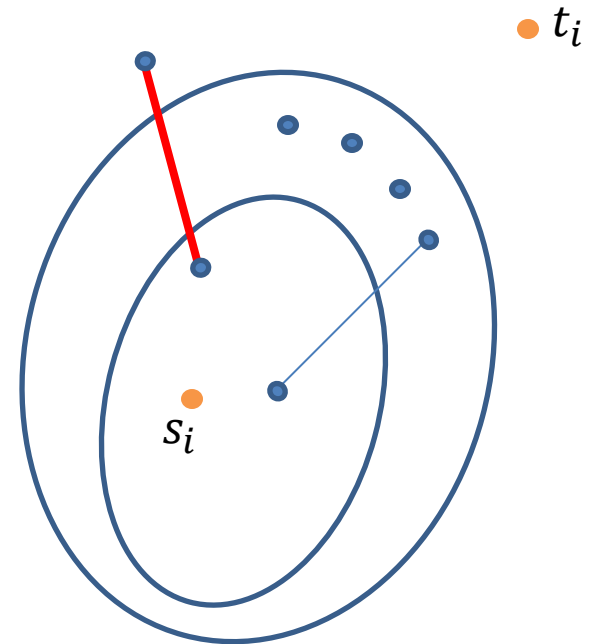
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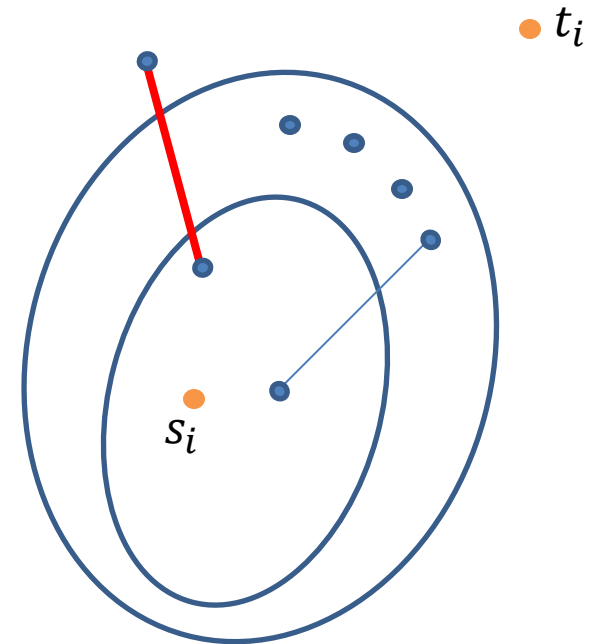
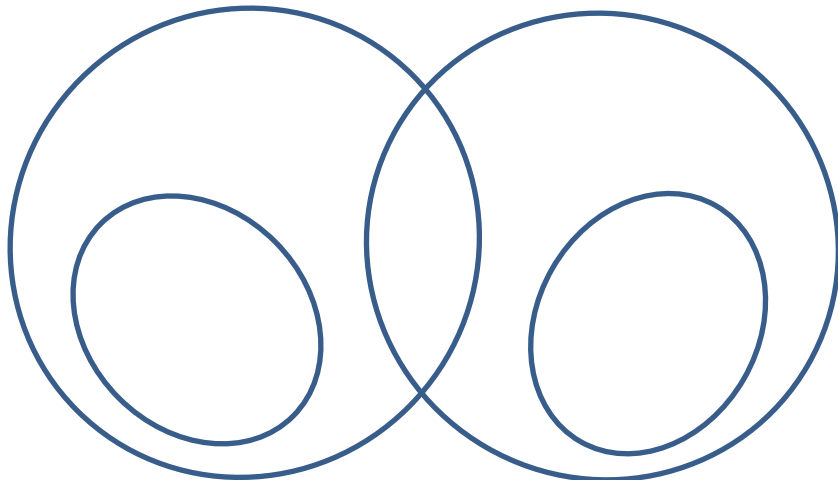
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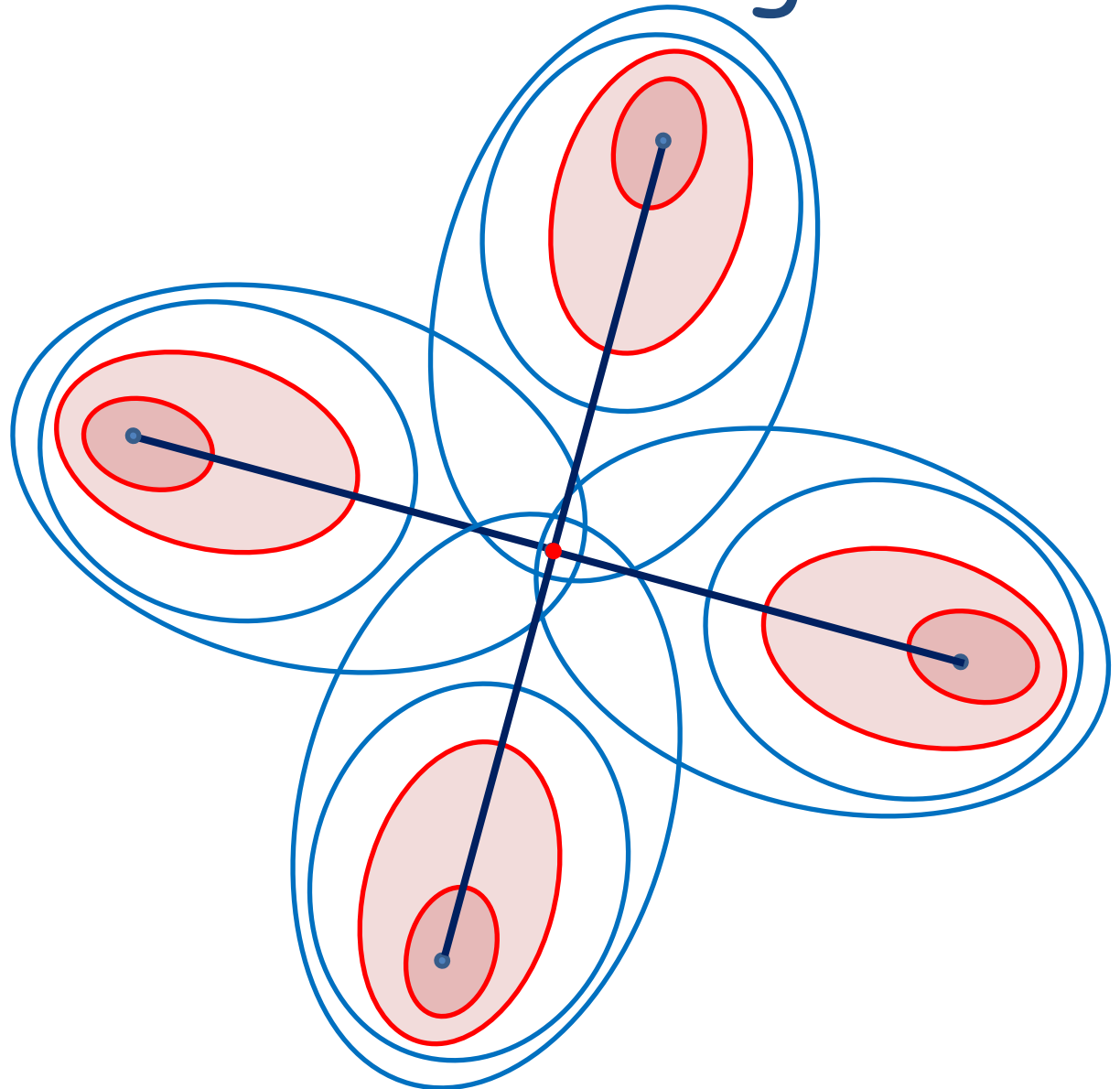
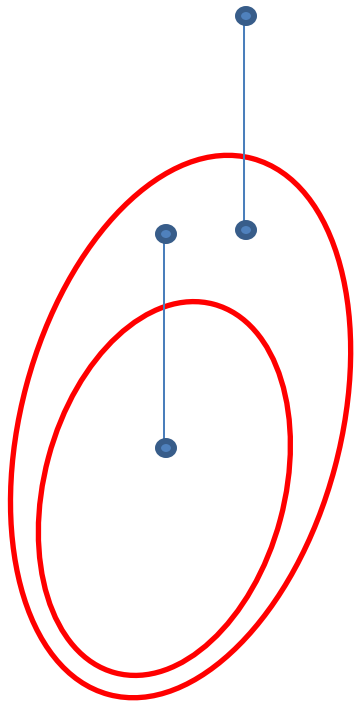
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What is the challenge?



How to solve?

Previous work considered abstract **skew-(bi)supermodular** function

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r is a **skew supermodular** function

$$r(A) + r(B) \leq \max\{r(A \cap B) + r(A \cup B), r(A \setminus B) + r(B \setminus A)\}$$

for any two sets A and B

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But, we analyze the iterative rounding method in less general setting

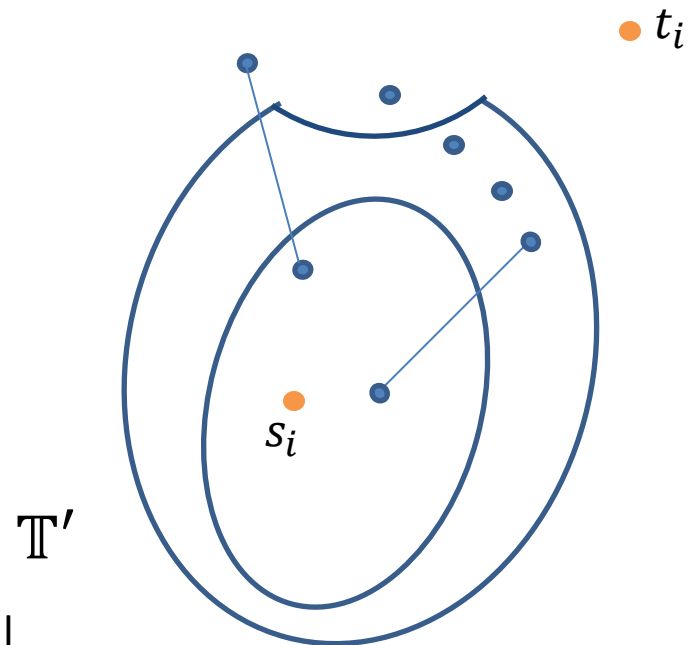
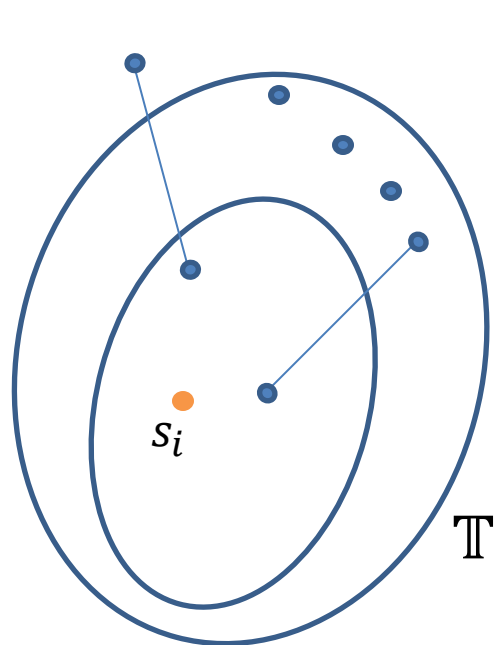
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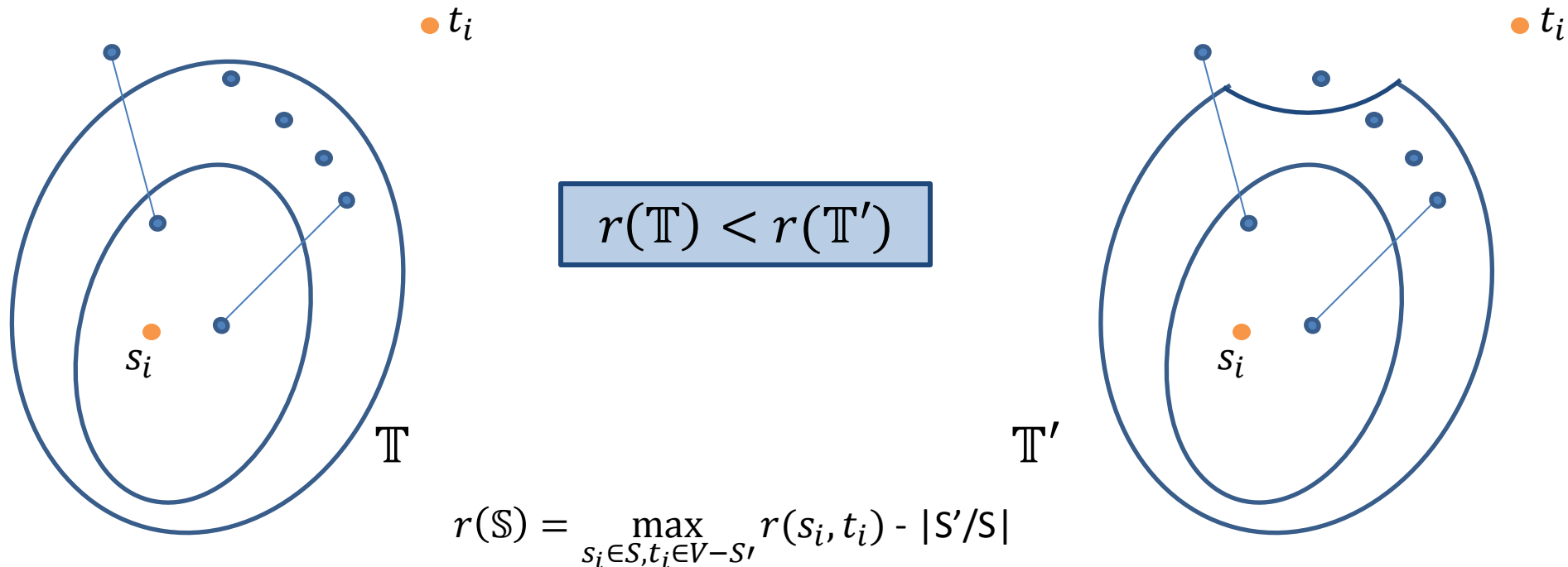
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$(3, 6b(v) + 5)$ -approximation for Deg-EC-SNDP

Degree bounded SNDP with node-connectivity

Problem	Previous work	Our result
Elem-SNDP	$(O(1), O(1)b(v) + O(k))$ [Fukunaga, Nutov and Ravi'13]	$(O(1), O(1)b(v))$
k-connectivity ($n \geq k^3$)	$(O(1), O(1)b(v) + O(k))$ [Fukunaga, Nutov and Ravi'13, Cheriyán and Vegh'13]	$(O(1), O(1)b(v))$
Rooted k-connectivity	$(O(1), O(1)b(v) + O(k))$ [Fukunaga and Ravi'12]	$(O(1), O(1)b(v))$
VC-SNDP	$(O(k^3 \log n), O(k^3 \log n)(b(v) + O(k)))$ [Fukunaga, Nutov and Ravi'13]	$(O(k^3 \log n), O(k^3 \log n)b(v))$
Directed rooted k-connectivity	$(O(1), O(1)b^+(v) + O(k))$ [Fukunaga and Ravi'12]	$(O(1), O(1)b^+(v))$

Future Directions

- Apply iterative rounding directly on VC-SNDP problem
- Analyze iterative rounding in more general setting, **skew-(bi)supermodular functions**

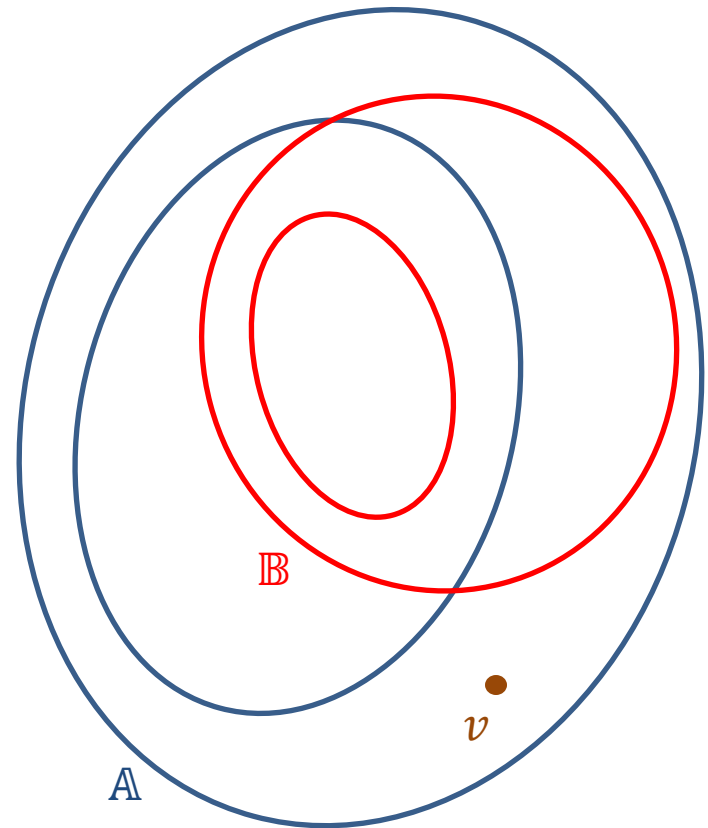
Thanks

Questions?

Main Observation

Relevant biset for a tight vertex

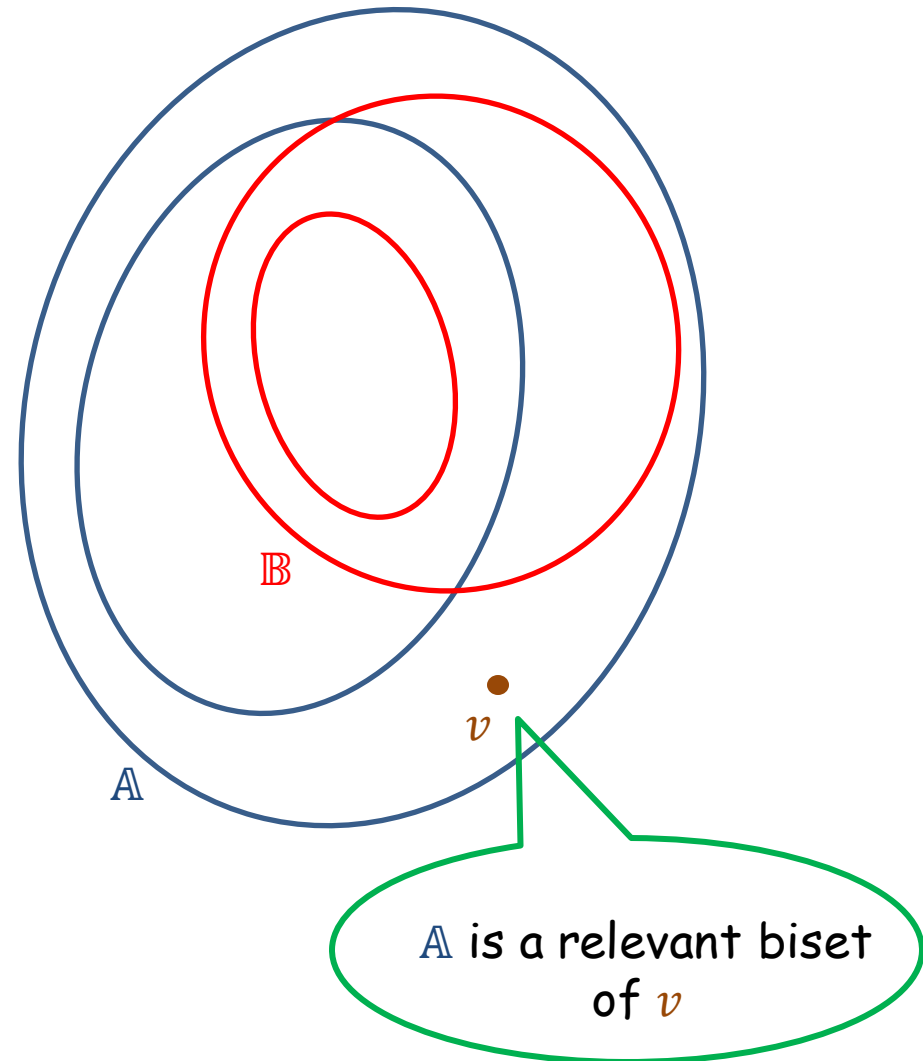
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- v is in $A' \setminus B'$; boundary of \mathbb{A} but not \mathbb{B}



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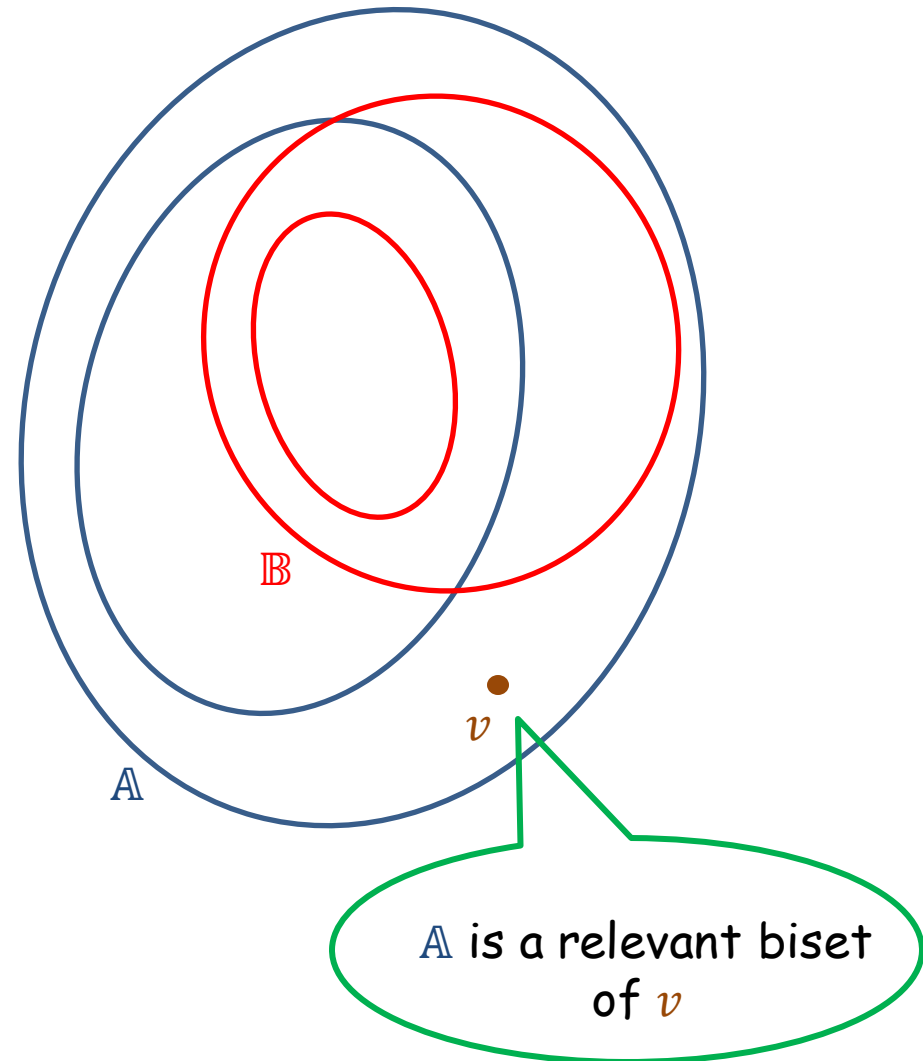
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Each tight vertex v , gives

2 tokens to each of its relevant bisets,
and 4 tokens to minimal bisets containing v

