6.1420 LECTURE 1

FINE-GRAINED ALGORITHMS AND COMPLEXITY

PERSONNEL

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6.1420 WORKLOAD

1. Three Problem sets: worth **60%** of grade, will come out about 2 weeks apart
   Can work with a partner; write up your own solutions.
2. Class Project: worth **40%** of grade
   Can work individually or with 1 partner
   Project proposal (1 page, **5%**): due **Nov 1**
   Progress report (2 pages, **5%**): due **Nov 14**
   (Also optional progress report due a week later, for feedback)
   *Final presentation (**10%**)*: during the last 1-2 weeks of class
   *Final report (5-10 pages, worth **20%**): due* **Dec 10**
3. Open problems session: Mondays

No exams!

WEBSITE AND PIAZZA

- **https://bit.ly/3QobcDG**
- Sign up for Piazza: link on the website, access code: refuteSETHplz
- Announcements will be made on both piazza and the website
- Assignments will be released on piazza

PLAN FOR THE DAY

What is this class about?

TWO MAJOR TOPICS

Fine-grained complexity and algorithms

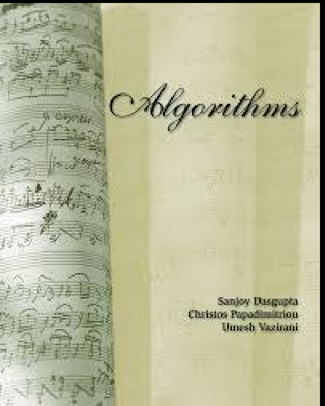
Today's lecture and first half of class

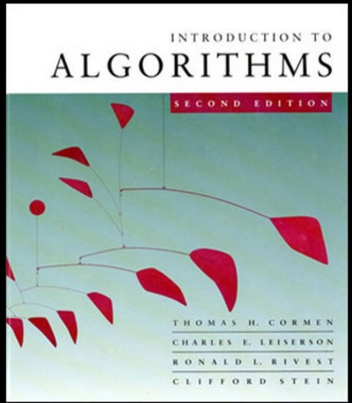
Fixed parameter complexity and algorithms

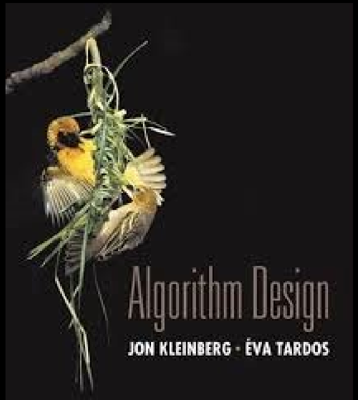
Second half of class

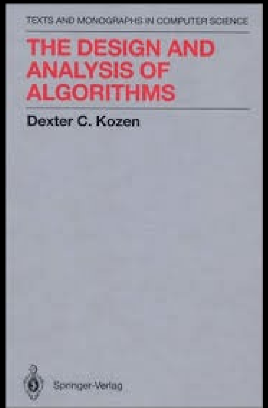
THE CENTRAL QUESTION OF ALGORITHMS RESEARCH

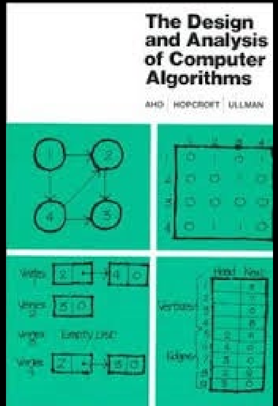
``How fast can we solve fundamental problems, in the worst case?''











etc.

HARD PROBLEMS

For many problems, the known techniques get stuck:

- Very important computational problems from diverse areas
- They have simple, often brute-force, textbook algorithms
- That are slow.
- No improvements in many decades!

Example 1

A CANONICAL HARD PROBLEM

k-SAT

Input: variables $x_1, \ldots, x_n$ and a formula

$F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ so that each C_i is of the form

$\{y_1 \vee y_2 \vee \ldots \vee y_k\}$ and $\forall i$, y_i is either x_t or $\neg x_t$ for some t.

Output: A boolean assignment to $\{x_1,\ldots,x_n\}$ that satisfies all the clauses, or NO if the formula is not satisfiable

Brute-force algorithm: try all 2^n assignments

Best known algorithm: $O(2^{n-(cn/k)} m^d)$ time for const c,d

Goes to 2^n as k grows.

Example 2

???

ANOTHER HARD PROBLEM: LONGEST COMMON SUBSEQUENCE (LCS)

Given two strings on n letters

ATCGGGTTCCTTAAGGG

ATTGGTACCTTCAGG

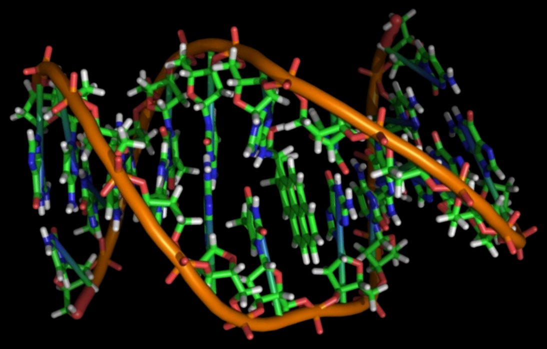
Find a subsequence of both strings of maximum length.

Algorithms:

Classical **$O(n^2)$** time

Best algorithm:
$O(n^2 / \log^2 n)$ time [MP'80]

Applications both in computational biology and in spellcheckers.



Solved daily on huge strings!

(Human genome: 3×10^9 base pairs.)

IN THEORETICAL CS, POLYNOMIAL TIME = EFFICIENT/EASY.

This is for a variety of reasons.

E.g. composing two efficient algorithms results in an efficient algorithm. Also, model-independence.

However, noone would consider an $O(n^{100})$ time algorithm efficient in practice.

If n is huge, then $O(n^2)$ is also inefficient.

WE ARE STUCK ON MANY PROBLEMS, EVEN JUST IN $O(N^2)$ TIME

No $N^{2-\varepsilon}$ time algorithms known for:

➨Many *string matching* problems:
Edit distance, Sequence local alignment, LCS, jumbled indexing …

***General form**: given two sequences of length n, how similar are they?*
All variants can be solved in $O(n^2)$ time by dynamic programming.

ATCGGGTTCCTTAAGGG
ATTGGTACCTTCAGG

WE ARE STUCK ON MANY PROBLEMS, EVEN JUST IN $O(N^2)$ TIME

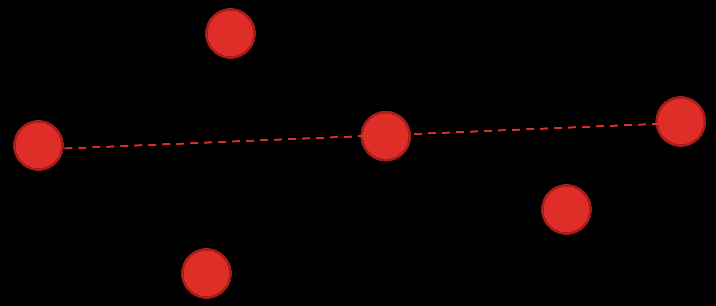
No $N^{2-\varepsilon}$ time algorithms known for:

➨Many *string matching* problems

➨Many problems in *computational geometry*: e.g

Given n points in the plane, are any three collinear?

A very important primitive!



WE ARE STUCK ON MANY PROBLEMS, EVEN JUST IN $O(N^2)$ TIME

No $N^{2-\varepsilon}$ time algorithms known for:

- Many *string matching* problems
- Many problems in *computational geometry*
- Many *graph problems* in sparse graphs: e.g.

Given an n node, O(n) edge graph, what is its diameter?

Fundamental problem. Even approximation algorithms seem hard!

WE ARE STUCK ON MANY PROBLEMS, EVEN JUST IN $O(N^2)$ TIME

No $N^{2-\varepsilon}$ time algorithms known for:

- Many *string matching* problems
- Many problems in *computational geometry*
- Many *graph problems* in sparse graphs
- Many other problems …

Why are we stuck?

Are we stuck because of the same reason?

PLAN FOR TODAY

- Traditional hardness in complexity
- A fine-grained approach
- Some simple results

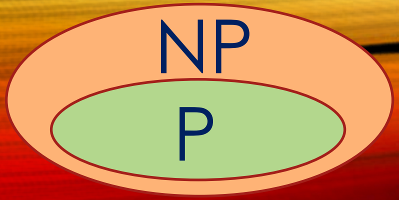
TIME HIERARCHY THEOREMS IN COMPLEXITY THEORY

For most natural computational models one can prove:

for any constant c, there *exist* problems solvable in $O(n^c)$ time but not in $O(n^{c-\varepsilon})$ time for any $\varepsilon > 0$.

It is completely unclear how to show that a *particular* problem in $O(n^c)$ time is not in $O(n^{c-\varepsilon})$ time for any $\varepsilon > 0$.

It is not even known if SAT is in linear time!



WHY IS K-SAT HARD?

N – size of input

It also does not apply to problems in P! Unless P=NP

Theorem [Cook, Karp'72]:

k-SAT is **NP-complete** for all $k \geq 3$.

NP-completeness addresses runtime, but it is too coarse-grained!

I.e. k-SAT is considered *hard because*

***"fast" algorithms for it imply "fast" algorithms for many important problems*.**

We'll develop a *fine-grained theory of hardness* that is conditional and mimics NP-completeness.

PLAN

- Traditional hardness in complexity
- A fine-grained approach
- Some simple results

FINE-GRAINED HARDNESS *IDEA*

Idea: Mimic NP-completeness

1. Identify key hard problems

2. Reduce these to all (?) problems believed hard

3. Hopefully form *equivalence classes*

CNF SAT IS CONJECTURED TO BE REALLY HARD

We will see these in detail in 2 lectures!

Two popular conjectures about SAT on n variables [IP01,CIP10]:

ETH (Exponential Time Hypothesis):

3-SAT requires $2^{\delta n}$ time for some constant $\delta > 0$.

SETH (Strong Exponential Time Hypothesis): For every $\varepsilon > 0$, there is a k such that k-SAT on n variables, m clauses cannot be solved in $2^{(1-\varepsilon)n}$ poly m time.

So we can use k-SAT as our hard problem and ETH or SETH as the hypothesis we base hardness on.

Strengthening of SETH [CGIMPS'16] suggests these are **not equivalent…**

Fix the model:
word-RAM with
O(log n) bit words

Given a set S of n vectors in $\{0,1\}^d$, for d = ω(log n) are there u, v 2 S with **u ¢v = 0**?

Hypothesis: Orthog. Vecs. requires $n^{2-o(1)}$ time.

[W'05]: SETH implies this hypothesis!

Easy O($\mathbf{n^2}$ d) time alg

Best known [AWY'15]: $\mathbf{n^{2 - \Theta(1 / \log (d/\log n))}}$

We will see these a lot!

Orthogonal vectors

Next 2 weeks

More key problems to blame

Hypothesis: APSP requires $n^{3-o(1)}$ time.

All pairs shortest paths: given an n-node weighted graph, find the **distance** between every two nodes.

Weeks 4-5

Given a set S of n integers, are there a, b, c 2 S with **a + b + c = 0**?

3SUM

APSP

Weeks 6-7

Easy O(n^2) time alg
[BDP'05]: ~$\mathbf{n^2}$ / $\log^2$ n time for integers
[Chan'18] : ~$\mathbf{n^2}$ / $\log^2$ n time for reals

Hypothesis: 3SUM requires $n^{2-o(1)}$ time.

Classical algs: O($\mathbf{n^3}$) time
[W'14]: $\mathbf{n^3}$ / exp(√ log n) time

FINE-GRAINED HARDNESS

Idea: Mimic NP-completeness

1. Identify key hard problems

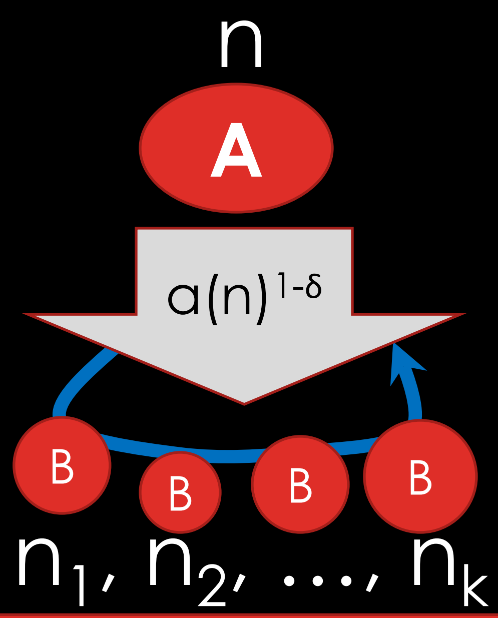
2. Reduce these to all (?) other hard problems

3. Hopefully form *equivalence classes*

FINE-GRAINED REDUCTIONS

Intuition: a(n),b(n) are the naive runtimes for A and B. A reducible to B implies that beating the naive runtime for B implies also beating the naive runtime for A.

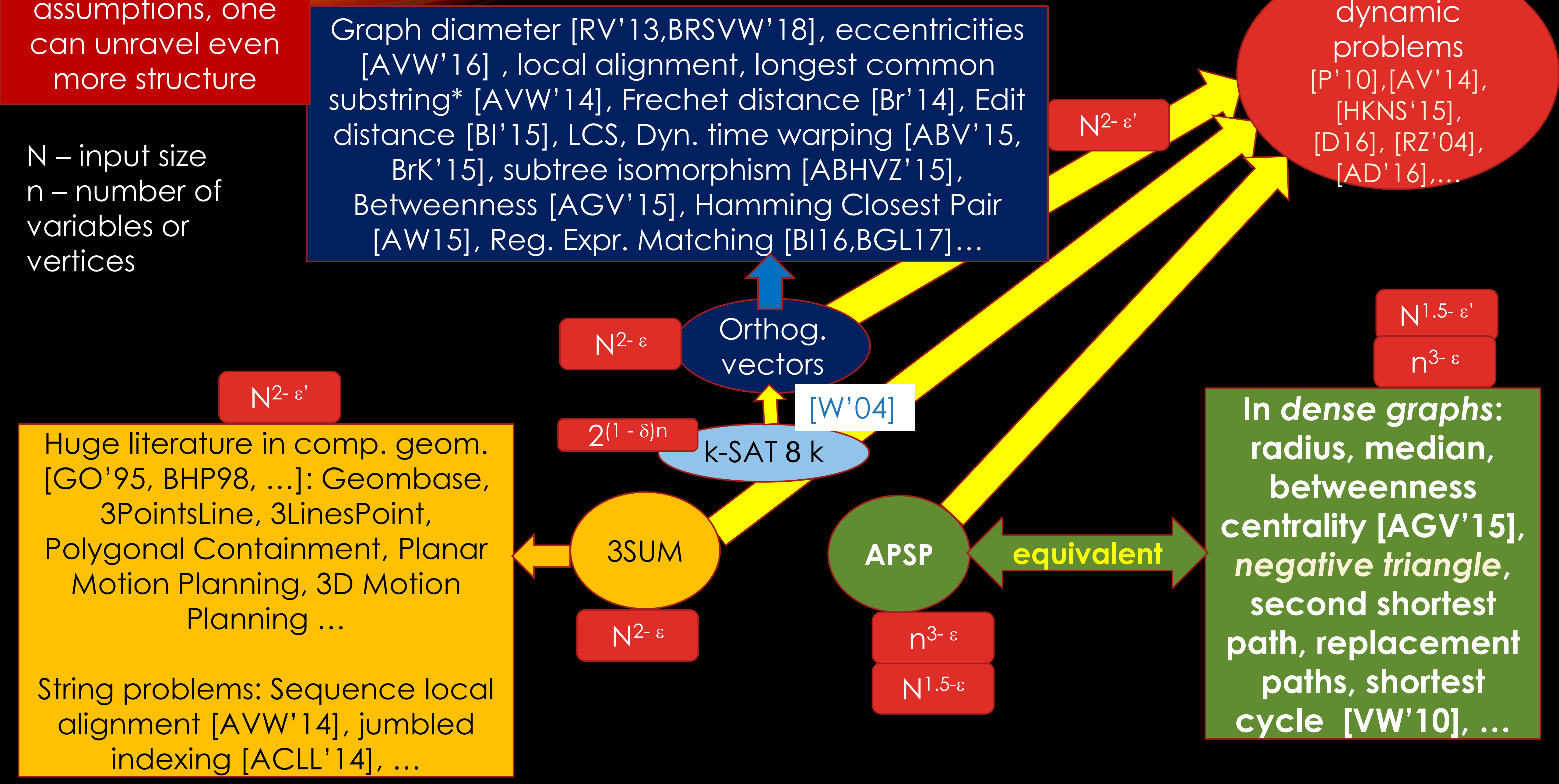
- A is (a,b)-reducible to B if
  for every $\varepsilon>0$ $\exists$ $\delta>0$, and an $O(a(n)^{1-\delta})$ time algorithm
  that adaptively transforms any A-instance of size n to B-instances of size $n_1,\ldots,n_k$ so that $\Sigma_i\, b(n_i)^{1-\varepsilon} < a(n)^{1-\delta}$.



- **If B is in $O(b(n)^{1-\varepsilon})$ time,**
  then A is in $O(a(n)^{1-\delta})$ time.
- Focus on exponents.
- We can build equivalences.

Don't worry! We will see many examples!

SOME STRUCTURE WITHIN P



PLAN

- Traditional hardness in complexity
- A fine-grained approach
- First reductions: from SETH

SETH

SETH: for every $\varepsilon > 0$, there is a k such that k-SAT on n variables, m clauses cannot be solved in $2^{(1-\varepsilon)n}$ poly m time.

If there is an $2^{(1-\varepsilon)n}$ poly m time algorithm for some $\varepsilon > 0$ that can solve SAT on CNF Formulas (for all k) on n variables and m clauses, then SETH is false.

FAST OV IMPLIES SETH IS FALSE [W'04]

F- CNF-formula on n vars, m clauses

E.g. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_2 \vee \neg x_4)$

Split the vars into V_1 and V_2 on $n/2$ vars each

E.g. $V_1 = \{x_1, x_2\}$, $V_2 = \{x_3, x_4\}$

OV: Given a set S of N vectors in $\{0, 1\}^d$, are there u, v 2 S with **u ¢v = 0**?

Given F, we want to create a set of vectors S in $\{0,1\}^d$ so that there is an orthogonal pair if and only if F is satisfiable and $|S| \sim 2^{n/2}$ and $d \sim m$.

For j=1,2 consider the **partial assignments** of V_j **: there are $2^{n/2}$ of them.**

E.g. for V_1: $\{[x_1 = 0, x_2 = 0], [x_1 = 0, x_2 = 1], [x_1 = 1, x_2 = 0], [x_1 = 1, x_2 = 1]\}$

FAST OV IMPLIES SETH IS FALSE [W'04]

F- CNF-formula on n vars, m clauses

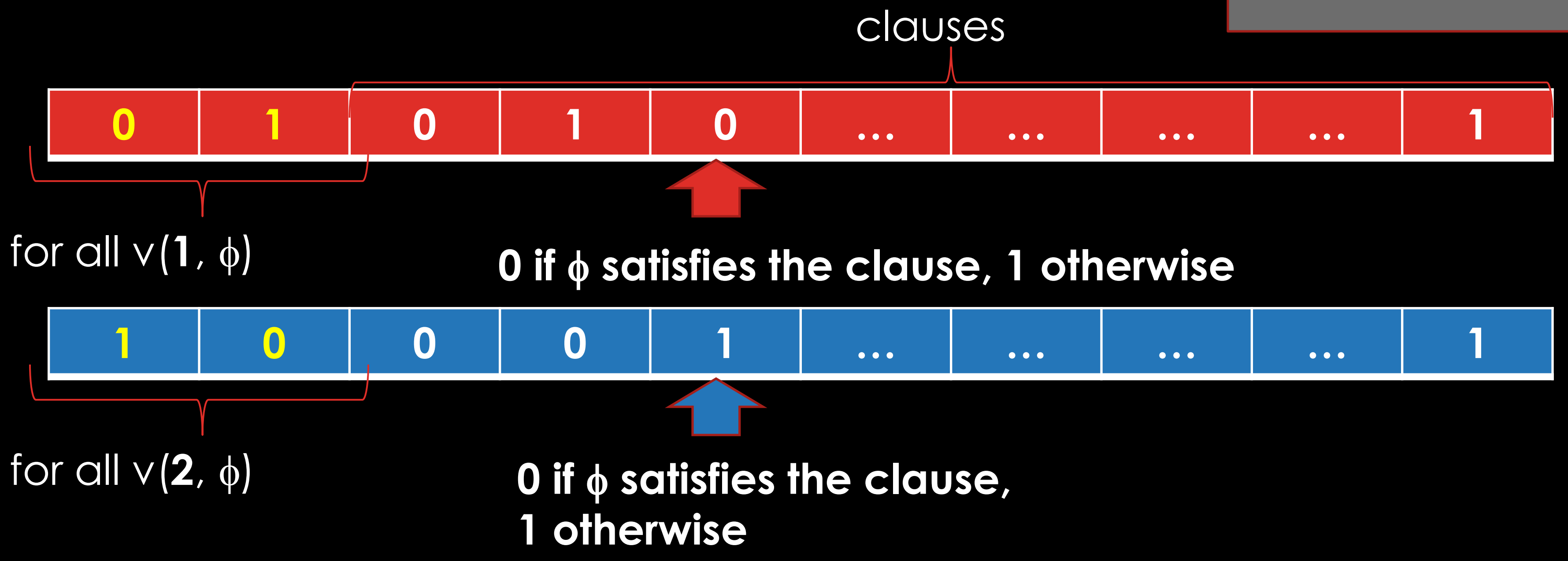
Split the vars into V_1 and V_2 on n/2 vars each

For j=1,2 and each **partial assignment** ϕ of V_j create (m+2) length vector $v(j, \phi)$:

E.g. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_3 \vee \neg x_4)$

$V_1 = \{x_1, x_2\}$, $V_2 = \{x_3, x_4\}$

$v(1, [x_1 = 0, x_2 = 0]) =$ [0, 1, 1, 0, 1]



FAST OV IMPLIES SETH IS FALSE

0	1	0	1	0	…	…	…	…	1

for all v(**1**, ϕ)

0 if ϕ satisfies the clause, 1 otherwise

1	0	0	0	1	…	…	…	…	1

for all v(**2**, ψ)

0 if ψ satisfies the clause,
1 otherwise

Claim: v(1, ϕ) ¢v(2, ψ) = 0 iff ϕ ⊙ ψ is a sat assignment.

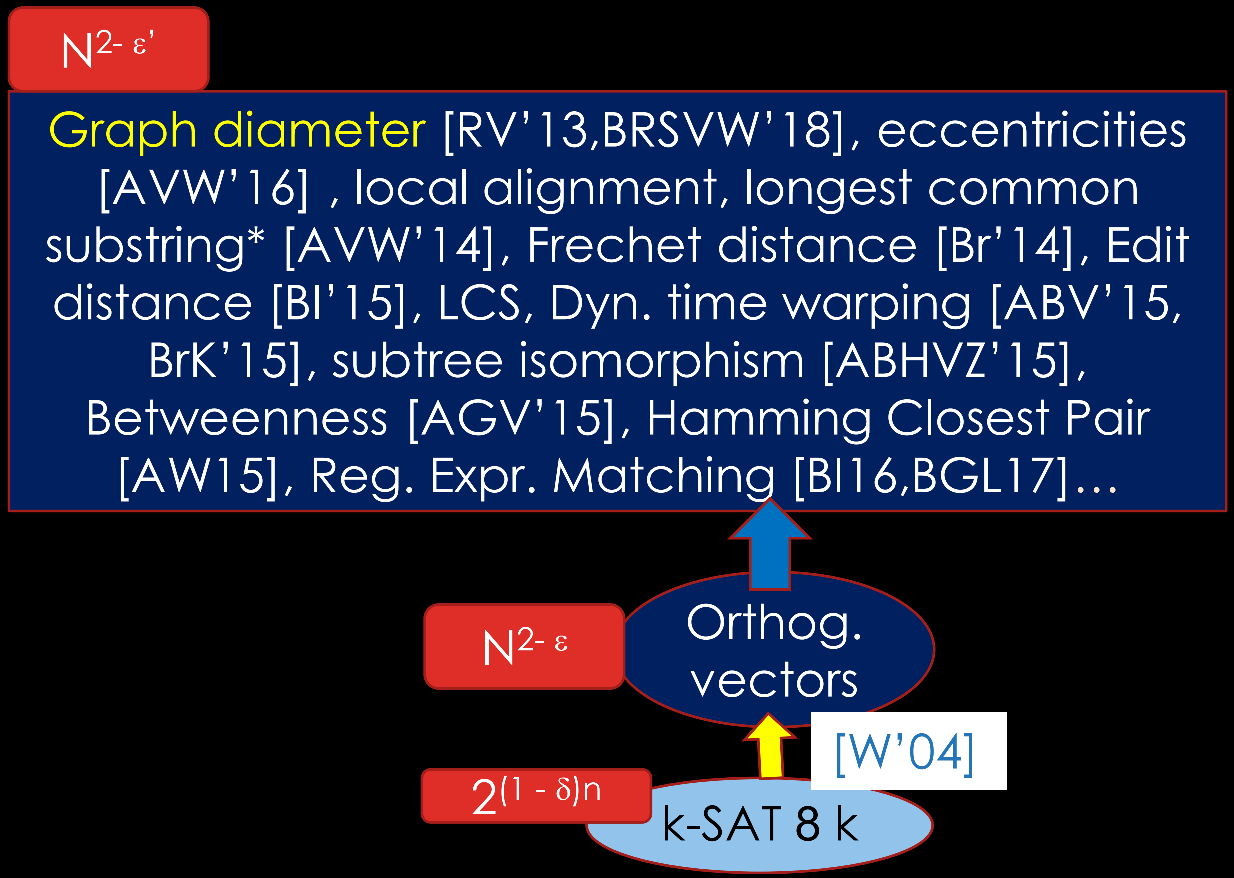
N = $2^{n/2}$ vectors of dimension $d = O(m)$ → an OV instance.

So $N^{2-\delta}$ $poly(d)$ time for OV for $\delta > 0$ implies $2^{n(1-\frac{\delta}{2})}$ $poly(m)$ time for SAT and SETH is false.

Diameter:

Given $G = (V, E)$, determine $D = \max_{u,v \in V} d(u, v)$.

$\frac{3}{2}$ – **Approximate Diameter**: output D’ such that $\frac{2D}{3} \leq D' \leq D$.



Say G has m edges, n vertices.

Using BFS: $O(mn)$ time Diameter.
Best known even in sparse graphs.

RV’13: 3/2-Approximate Diameter in $\tilde{O}\left(m^{\frac{3}{2}}\right)$ time – better than mn in sparse graphs!

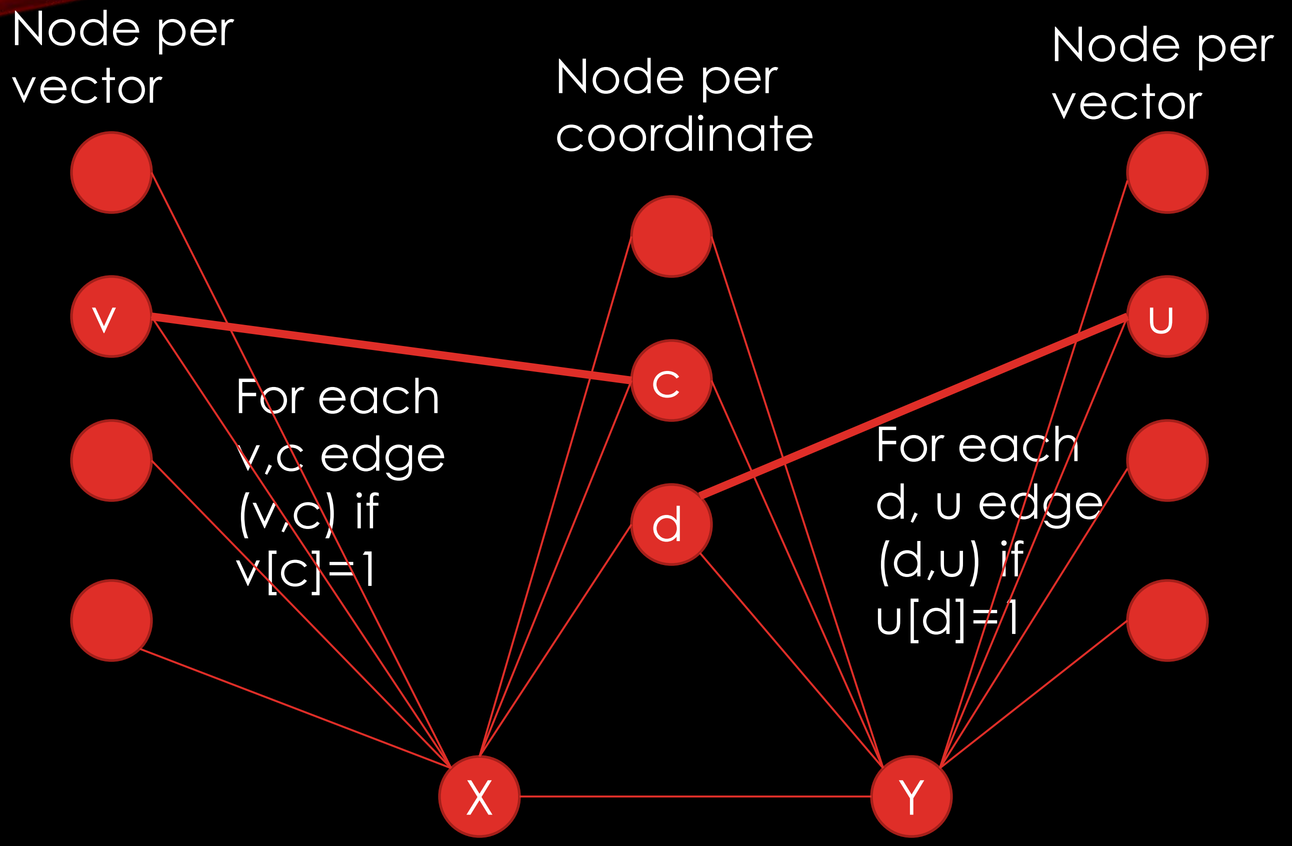
We’ll show 3/2$-\epsilon$ – Diameter for $\epsilon > 0$ requires $mn^{1-o(1)}$ time under SETH.

Hard: distinguishing between Diameter 2 or 3 in sparse graphs.

Reduce from OV on n vectors; due to "Sparsification Lemma" can assume dimension is $d = O(\log n)$…

[RV'13]

DIAMETER 2 OR 3



Diameter is 3 if exists orthogonal pair, and is 2 otherwise.

Thm: Diameter 2 or 3 in $O(m^{2-\varepsilon})$ time implies $O(n^{2-\delta})$ time for OV and hence SETH is false.

Any two vector nodes from the same side are at dist 2.

Any coordinate is at dist 2 from everyone, X and Y are at dist 2 from everyone.

Two vectors u and v from different sides are at **dist 2** if exists a c with **u[c]=v[c]=1**, and at **dist 3** otherwise.

Graph has $O(n)$ nodes and since $d = O(\log n)$, $m = \tilde{O}(n)$ edges

SEE YOU NEXT TIME!

- Check Piazza and the website for the lecture notes.
- PS: If there's any related topic you'd like more lecture notes to read (e.g., NP-completeness) please let us know via piazza!
  (You can also email, but it's better if other students can see your questions too, so they can upvote them!)