6.890 Lecture 6 APSP variants: Earliest Arrivals, Bottlenecks
Scribe: Vaggos Chatziafratis **Date:** September 28, 2021
Edited by: Nicole Wein

Today we will present and solve two more variants of All Pairs Shortest Paths (APSP) in $O(n^{3-\delta})$ time for some constant $\delta > 0$. In doing so, we will also introduce some more matrix products, namely the $(\max, \min)$-product, the $(\min, \leq)$ product and the dominance product.

1 Earliest Arrivals

The first variant of APSP we will study is the Earliest Arrivals problem. We are given a set V consisting of n airports and a set F of n flights. Each flight $f \in F$ consists of a a source airport $s \in V$, a destination airport $t \in V$, a departure time, and an arrival time.

Definition 1.1. *A valid itinerary from s to t is a sequence of flights $f_1, \dots f_k$ such that, for all $i \in \{1, \dots, k\}$, $source(f_{i+1}) = destination(f_i)$ and $departure(f_{i+1}) \geq arrival(f_i)$.*

The All-Pairs Earliest Arrivals (APEA) problem is to compute, for all airports $u, v \in V$, the earliest arrival time over all valid itineraries. This problem has a natural graph interpretation. Consider a bipartite graph $G = (V \cup F, E)$. For each flight $f \in F$, we add a directed edge $(source(f), f)$ to E with weight $departure(f)$. Then, we add another directed edge $(f, destination(f))$ with weight $arrival(f)$.

On this graph, a valid itinerary is a $s \to t$ path such that all of the edges form a nondecreasing sequence, and the arrival time is given by the last edge weight. Therefore, APEA is equivalent to finding, $\forall s, t \in V$, the minimum last edge weight over all nondecreasing $s \to t$ paths.

Let's consider the special case of APEA restricted to 2-hop paths. Consider two nodes x and y. The best 2-hop path between them has arrival time which is the minimum of $w(z, y)$ over all mid-points z such that both (x, z) and (z, y) are edges and $w(x, z) \leq w(z, y)$. This gives rise to a new matrix product.

Definition 1.2. *Let A, B be $n \times n$ matrices. The $(\min, \leq)$ product of A and B, denoted $A \circledcirc B$ is given by*

$$(A \circledcirc B)(i, j) = \min_k \{B(k, j) \mid A(i, k) \leq B(k, j)\}$$

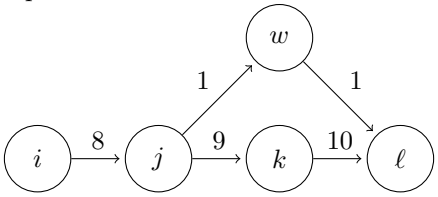
or ∞ if no such k exists.

If we define the adjacency matrix A of G in the natural way,

$$A(i, j) = \begin{cases} w(i, j) & (i, j) \in E \\ 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

then we find that $(A \circledcirc A)(i, j)$ is the minimum last edge weight over paths of length 2. Iterating this relationship, we find that $(\underbrace{A \circledcirc \cdots \circledcirc A}_{\ell - 1 \text{ times}}) \circledcirc A(i, j)$ is the minimum last edge over all paths of length ℓ.

We must be careful, however, because the $(\min, \leq)$ product is not associative in general, as the following example demonstrates. Consider the following graph.



Observe that $(A \circledcirc A) \circledcirc A(i, \ell) = 10$, but $A \circledcirc (A \circledcirc A)(i, \ell) = \infty$. Consequently, we cannot simply use successive

squaring to solve the Earliest Arrivals problem. Instead, our approach will be to compute Earliest Arrivals for "short paths" and use the random sampling technique developed in previous lectures to handle "long paths." For paths that have at most s hops (for a parameter s), we iterate the $\circledcirc$ product with the adjacency matrix A on the right s times. To handle paths on $> s$ hops, we sample $O(n/s \log n)$ nodes S - these hit for each u, v with long hop best path $\pi(u, v)$, one of the nodes on $\pi(u, v)$, with high probability. Then we utilize an algorithm that given any node s can compute All-Pairs Earliest arrivals for G but restricted only to those nondecreasing paths that pass through s. (We will show in the homework that this latter problem can be solved in $O(n^2 \log n)$ time.) We use this algorithm for each node in S and then for each u, v we take the minimum value obtained in the short-paths part and in all $\tilde{O}(n/s)$ executions that handle the long-path part.

Consider the algorithm below.

Algorithm 1: Earliest Arrivals(G)

```
Form adjacency matrix A
Set D = A
  for i := 1 to s do
    Compute D = D ⊚ A
  end for
  Compute a random sample, S, of size c * (n/s) log n
  for all x ∈ S do
    Compute All Pairs Earliest Arrivals for paths through x
  end for
  for all i, j ∈ V do
    EA(i, j) = min_{x∈S} min last edge weight over valid itineraries of the form i → x → j
    EA(i, j) = min{EA(i, j), D(i, j)}
  end for
  Return EA
```

Lemma 1.1. *If the* $(\min, \leq)$ *product of* $n \times n$ *matrices can be computed in* $O(n^c)$ *time, then we can solve APEA in* $O(n^{\frac{3+c}{2}})$ *time.*

Proof of Lemma 1.1. Using the algorithm sketched above, we obtain a runtime of $O(\frac{n^3}{s} + s(n^c))$. Optimizing over s, we set $s = n^{\frac{3-c}{2}}$ and obtain a total runtime of $O(n^{\frac{3+c}{2}})$, as required. □

2 All Pairs Bottleneck Paths

Let graph $G = (V, E)$ be a graph with edge weights given by $w : E \to \mathbb{Z}$.

Definition 2.1. *Given a path* p *in* G*, its bottleneck edge is the edge of minimum weight.*

Definition 2.2. *The All Pairs Bottleneck Paths problem (APBP) is to find, for all pairs* $u, v \in V$*, the maximum bottleneck weight over all* $u \to v$ *paths.*

For example, imagine the weight of each edge represents the height of a tunnel. Then we want to find the maximum height of a truck that can get from s to t and fit through all the tunnels.

In order to tackle this problem, we need to define another matrix product.

Definition 2.3. *Let* A *and* B *be* $n \times n$ *matrices. The* $(\max, \min)$ *product of* A *and* B*, denoted* $A \circledast B$ *is given by*

$$(A \circledast B)(i, j) = \max_k \min\{A(i, k), B(k, j)\}$$

Observe that that $(\max, \min)$ product is precisely the bottleneck path problem in graphs with diameter 2. It is left as an exercise to verify that $\circledast$ is associative and is in fact a matrix product defined over a semiring. Thus, $A \circledast A$ gives the maximum bottleneck for length 2 paths, we can solve All Pairs Bottleneck Paths (APBP) using successive squaring. This gives us the following lemma.

Lemma 2.1. *If the* $(\max, \min)$ *product of two* $n \times n$ *matrices can be computed in* $\tilde{O}(n^c)$ *time, then we can solve All Pairs Bottleneck Paths in* $\tilde{O}(n^c)$ *time.*

Using the general theorem of Fischer and Meyer about transitive closure of matrices over semirings, one can also get that the extra log due to the successive squaring isn't even necessary so that APBP and $(\max, \min)$-product are equivalent.

Let us now see that computing $\circledast$ is equivalent to two $\circledcirc$ product computations. This will give us that APBP can be computed in truly subcubic time provided we get a truly subcubic algorithm for $\circledcirc$ product.

Lemma 2.2. *If there is an* $O(n^c)$ *algorithm for computing* $(\min, \leq)$ *products, there is an* $O(n^c)$ *algorithm for computing* $(\max, \min)$ *products.*

Proof. Consider the matrix product defined by $(A \circledcirc B)(i,j) = \max_k\{A(i,k) \mid A(i,k) \leq B(i,k)\}$. Note that this product is in fact a $(\min, \leq)$ product. In particular, it is the product $-B \circledcirc -A$ obtained by negating all of the entries $a_{i,j}$ in A and $b_{i,j}$ in B and then swapping matrices A and B, i.e. $(A \circledcirc B)(i,j) = -(-B \circledcirc -A)(i,j)$. Using this product, we can compute

$$(A \circledast B)(i,j) = \max\{(A \circledcirc B)(i,j), (B \circledcirc A)(i,j)\}.$$

Therefore, we can compute $A \circledast B$ using two $(\min, \leq)$ computations, as required. □

An interesting note is that there is no known reduction in the other direction, so it is possible that there is a faster algorithm for (max,min) product than for $(\min, \leq)$ product.

By the above discussion, we can solve both the All Pairs Earliest Arrivals problem and the All Pairs Bottleneck Path problem with a fast algorithm for computing $(\min, \leq)$ products. The rest of this writeup is dedicated to finding such an algorithm.

3 A Fast Algorithm for Computing $(\min, \leq)$ Products

We will use another special matrix product in our algorithm for computing $(\min, \leq)$.

Definition 3.1. *The dominance product of* $n \times n$ *matrices* A *and* B*, denoted* $A \odot B$*, is given by*

$$(A \odot B)(i,j) = |\{k \mid A(i,k) \leq B(k,j)\}|$$

Theorem 3.1. *(Matoušek'91) The dominance product of two* $n \times n$ *matrices can be computed in* $O(n^{\frac{3+\omega}{2}})$ *time.*

Theorem 3.2. *If dominance product can be computed in* $O(n^d)$ *time, then the* $(\min, \leq)$ *product can be computed in* $O(n^{\frac{3+d}{2}})$ *time.*

Assuming 3.1, we first prove 3.2.

Proof of Theorem 3.2. Let A, B be two $n \times n$ matrices. We will compute $A \circledcirc B$ as follows.

1. Sort each column j of matrix B
2. Fix parameter p. Partition each sorted column into $\frac{n}{p}$ consecutive buckets of p elements each. Name the buckets so that for all buckets $b \leq b'$, $\forall B(i,j)$ in bucket b of column j, and $\forall B(\ell,j)$ in bucket b' of j, we have $B(i,j) \leq B(\ell,j)$.

3. For each $b \in \{1, \ldots, \frac{n}{p}\}$, create an $n \times n$ matrix B_b such that

$$B_b(i,j) = \begin{cases} B(i,j) & \text{if } B(i,j) \text{ in bucket } b \text{ of column } j \\ -\infty & \text{otherwise} \end{cases}$$

4. Compute for all buckets b, $A \odot B_b$, which is

$$(A \odot B_b)(i,j) = \begin{cases} \neq 0 & \text{if } \exists k \text{ such that } B_b(k,j) \neq -\infty \text{ and } A(i,k) \leq B(k,j) \\ 0 & \text{otherwise} \end{cases}$$

5. For all i,j determine $b_{i,j}$ = smallest b such that $(A \odot B_b)(i,j) \neq 0$. This is equivalent to

$$\min\{B[k,j] \mid B(k,j) \text{ in bucket } b(i,j) \text{ and } A(i,k) \leq B(k,j)\}.$$

   Therefore, we can use brute force, as follows. For all i,j examine each $B(k,j)$ in bucket $b_{i,j}$ of j, compare it with $A(i,k)$ and output the minimum $B(k,j)$ for which $A(i,k) \leq B(k,j)$. Observe that this is $(A \circledcirc B)(i,j)$.

The running time of this algorithm is dominated by computing the dominance product in step 4 and brute force in step 5. Using 3.1, we can compute dominance product in $O(n^d)$ time. Therefore, it takes $O(\frac{n^{d+1}}{p})$ time to compute the required $\frac{n}{p}$ dominance products. The brute force step takes $O(n^2 p)$ time. Choosing $p = n^{\frac{d-1}{2}}$, we obtain a total runtime of $O(n^{\frac{3+d}{2}})$, as desired. □

It remains to prove 3.1

Proof of Theorem 3.1. Let A, B be $n \times n$ matrices. We compute $A \odot B$ as follows.

1. For all j, sort the set of entries of column j of A and row j of B together. This produces a list of $2n$ elements.
2. Partition this list into buckets of p elements each. There are $\frac{2n}{p}$ buckets for each j.
3. For all $b \in \{1, \ldots, \frac{2n}{p}\}$, create $n \times n$ matrices

$$A_b(i,j) = \begin{cases} 1 & \text{if } A(i,j) \text{ is in bucket } b \text{ of } j \\ 0 & \text{otherwise} \end{cases}$$

$$B_b(j,k) = \begin{cases} 1 & \text{if } \exists b' > b \text{ such that } B(j,k) \text{ is in bucket } b' \text{ of } j \\ 0 & \text{otherwise} \end{cases}$$

4. For distinct buckets b, compute the integer matrix product

$$(A_b B_b)(i,j) = |\{k \mid A(i,k) \in b, A(i,k) \leq B(k,j), \text{ and } B(k,j) \notin b\}$$

   We handle identical buckets b using brute force search. For all i,j and buckets b, compare $A(i,k)$ with all $B(k,j)$ in the same bucket as $A(i,k)$ and update the sum in the output.

The brute force step requires $O(n^2 p)$ time. Then, we require $\frac{n}{p} n^\omega$ time to perform matrix multiplications. We minimize p by taking $p = n^{\frac{3-\omega}{2}}$ to obtain a final running time of $O(n^{\frac{3+\omega}{2}})$, as desired. □

It is an open problem whether $(\min, \leq)$ product can be computed in $O(n^\omega)$ time.

4 Conclusions

In this lecture we saw several different matrix products that are useful for many applications. The $(\min, \leq)$ product (which is not associative) is useful when searching for nondecreasing paths and has applications in the All Pairs Earliest Arrivals problem (APEA). The (max, min) product is used when searching for All Pairs Bottleneck Paths (APBP). Finally, we defined the dominance product. Using all these, we concluded, similar to the node-weighted APSP from last lecture, that APEA and APBP are in truly subcubic time. However, it remains a major open question to find a truly subcubic algorithm for general APSP.