1 Sparsification

Recall the Exponential Time Hypothesis (ETH): \( \exists \delta > 0 \) such that 3-SAT requires \( 2^{\delta n} \) time.

ETH is a strengthening of \( P \neq NP \): not only does 3-SAT require super-polynomial time, but it requires exponential time.

Now, when you saw NP-hardness reductions in prior algorithms classes, there were many of them from \( \text{3-SAT} \) to \( \text{IS} \). Could we use these, and by replacing \( P \neq NP \) as an assumption by ETH, obtain that a variety of other problems require exponential time?

Let us explore this question by revisiting the standard reduction from \( \text{3-SAT} \) to Independent Set.

The standard reduction from \( \text{3-SAT} \) to Independent Set. Recall the Independent Set (IS) Problem: Given a graph \( G = (V, E) \) and an integer \( k \), is there a subset \( I \subseteq V \) with \( |I| \geq k \) so that for every \((u, v) \in E\), either \( u \notin I \) or \( v \notin I \) (or both), i.e. \( I \) is independent.

Let us recall how to reduce \( \text{3-SAT} \) to IS. Given a 3-CNF formula \( F \) with \( m \) clauses and \( n \) variables, we produce a graph as follows. For every clause \( C_j \) with literals \( \ell_1, \ell_2, \ell_3 \), we create three nodes, one for each literal. Let’s call the node corresponding to literal \( \ell_k \) appearing in clause \( C_j \), \( v(C_j, \ell_k) \). We make the three nodes corresponding to each clause into a triangle: connect every two vertices corresponding to the same clause by an edge.

Then, consider every variable \( x_k \) of \( F \). If \( x_k \) appears as \( x_k \) in clause \( C_j \) and as \( \neg x_k \) in some other clause \( C_t \), then connect \( v(C_j, x_k) \) and \( v(C_t, \neg x_k) \) by an edge.

In other words, there are two types of edges: those connecting nodes for the same clause, and those connecting nodes for the same variable occurring negated and not-negated.

We will show that this graph \( G \) contains an independent set of size \( m \) if and only if \( F \) is satisfiable.

Now, suppose that \( F \) is satisfied by some assignment \( \sigma \). Assignment \( \sigma \) selects (at least) one literal per clause to set to true. Let \( \ell(j) \) denote that literal for clause \( C_j \). Consider \( I = \{ v(C_1, \ell(1)), \ldots, v(C_m, \ell(m)) \} \). \( I \) has \( m \) vertices of \( G \), one for each clause. Let’s see why \( I \) is independent. There are no edges of the first type since there are no two nodes for the same clause. Suppose then that there is an edge in \( G \) between \( v(C_i, \ell(i)) \) and \( v(C_j, \ell(j)) \). But then, \( \ell(i) = \neg \ell(j) \) since it must be an edge of the second type. This is a contradiction since the assignment \( \sigma \) cannot set both \( \ell(j) \) and \( \neg \ell(j) \) to true.

Now suppose that \( G \) has an independent set \( I \) of size \( m \). \( I \) can have at most one vertex \( v(C_j, \circ) \) for each clause \( C_j \) since any two vertices for the same clause are connected by an edge. Since \( I \) has size \( m \), it must actually have exactly one vertex per clause. Let \( v(C_j, \ell(j)) \) be the node for clause \( C_j \). Then we create an assignment for the variables of \( F \) as follows: for every \( \ell(j) \), let the assignment be such that \( \ell(j) \) is set to true. Notice that since there are no \( \ell(j) \) and \( \ell(i) \) for which \( \ell(j) = \neg \ell(i) \), the assignment to the variables corresponding to the literals \( \ell(j) \) are consistent. To complete this to a full assignment, assign 1 to all variables that do not appear in the literals \( \ell(j) \). We get an assignment that sets at least one literal to true in each clause, and hence \( F \) is satisfiable.

An example IS instance is shown in Figure 1. The triangles in the figure correspond to the clauses.

Independent Set requires exponential time, under ETH. Let’s attempt to use the above reduction to show that IS requires exponential time under ETH. Our supposed proof would proceed by contradiction. Suppose that for every \( \delta > 0 \), there is an \( O(2^{\delta N}) \) time algorithm for IS in \( N \)-node graphs. Now, given a 3-CNF \( F \) on \( n \) variables and \( m \)
\[ F = (x_1 \lor \neg x_2 \lor y_1) \land (\neg y_1 \lor x_3 \lor \neg x_4) \land (\neg x_1 \lor x_2 \lor y'_1) \land (\neg y'_1 \lor \neg x_3 \lor x_4). \]

Figure 1: The reduction from 3-SAT to IS for a particular formula \( F \). The corresponding Independent Set is shown, together with the corresponding satisfying assignment of \( F \).

clauses, we want to use this algorithm to solve \( F \) in \( O(2^{\varepsilon n} \cdot \text{poly}(m)) \) time for all \( \varepsilon > 0 \). We pick our favorite \( \varepsilon > 0 \) and we want to pick the \( \delta \) as a function of \( \varepsilon \) for the \( 2^{\delta N} \) time IS algorithm.

We use the reduction to form a graph \( G \) from \( F \). The number of nodes \( N \) of \( G \) is \( 3m \) as we have three nodes per clause. So for the \( \delta \) we pick, we would be solving 3-SAT in time \( O(2^{3\delta m}) \). Now since \( m \) can be as large as \( n^3 \), there can be no constant \( \delta \) that we can pick to get an \( 2^{\delta N} \) time algorithm for 3-SAT!

What we would like is a reduction that produces an IS instance graph on \( O(n) \) vertices instead of \( O(m) \). Then, our approach would work. Suppose that we could get a graph \( G \) out of a 3-SAT formula \( F \) so that \( G \) has \( N \leq en \) nodes when \( F \) has \( n \) variables. Then, for every \( \varepsilon > 0 \), we set \( \delta = \varepsilon/c > 0 \) and use the supposed \( O(2^{\delta N}) \) time IS algorithm to solve \( F \) in time \( O(2^{\varepsilon n}) \) due to our choice of \( \delta \).

One way to achieve such a reduction would be to sparsify 3-SAT, i.e. reduce 3-SAT on \( n \) variables and an arbitrary number of clauses to 3-SAT on \( O(n) \) variables and \( O(n) \) clauses. Then, we can just use the reduction we already have. Such a reduction is not known, and in fact, if such a reduction could be carried out in polynomial time, then coNP would be in NP/poly, something that we do not believe is true.

Fortunately, an almost as good sparsification is known. It is an algorithm that takes an arbitrary \( k \)-CNF (for any \( k \geq 3 \)) formula on \( n \) variables (possibly with a huge number of clauses) and produces a family of formulas with a linear number of clauses. Specifically:

**Theorem 1.1** Let \( \varepsilon > 0 \), \( k \geq 3 \) constant. There is a \( 2^{\varepsilon n} \cdot \text{poly}(n) \) time algorithm that takes a \( k \)-CNF \( F \) on \( n \) variables and produces \( F_1, \ldots, F_{2^n}, 2^n \) \( k \)-CNFs such that \( F \) is satisfied if and only if \( \bigvee_i F_i \) is satisfied and each \( F_i \) has \( n \) variables and \( n \cdot (\frac{k}{\varepsilon})^{O(k)} \) clauses. In fact, each variable is in at most \( \text{poly}(\frac{1}{\varepsilon}) \) clauses, and the \( F_i \) are over the same variables as \( F \).

We won’t prove the theorem above, but at the end of the notes we will give some intuition for the proof.

Now, let’s see how to use Theorem 1.1 together with the 3-SAT to IS reduction to show that IS requires exponential time under ETH.

Assume that IS on \( N \) node graphs can be solved in \( O(2^{\delta N}) \) time for all \( \delta > 0 \). Suppose that the term \( (\frac{k}{\varepsilon})^{O(k)} \) from Theorem 1.1 is \( \frac{1}{C} \) (i.e. \( C \) is the constant in the big-O). Consider any \( \gamma > 0 \) and let \( \delta = \gamma^{3C+1}/(6 \cdot 27C \cdot 8C) > 0 \) and \( \varepsilon = \gamma/2 > 0 \). Run the reduction from Theorem 1.1 for that choice of \( \varepsilon \), obtaining \( F_1, \ldots, F_{2^n} \) on \( n(3/\varepsilon)^{3C} \)
clauses each.

Let \( f(\varepsilon) = (3/\varepsilon)^{3C} \) for brevity. For every \( i \), reduce \( F_i \) to IS using the standard reduction, obtaining a graph \( G_i \) on \( 3nf(\varepsilon) \) nodes. Then we solve IS on \( G_i \) in time \( O(2^{3nf(\varepsilon)}) \). The total runtime over all \( i \) is \( O(2^{n(3f(\varepsilon)+\varepsilon)}) \).

Consider the coefficient in front of \( n \) in the exponent of this runtime:

\[
\varepsilon + 3\delta f(\varepsilon) = \gamma/2 + 3 \cdot (3 \cdot 2/\gamma)^{3C} \cdot \gamma^{3C+1}/(6 \cdot 2^C \cdot 8^C) = \\
\varepsilon/2 + \gamma/2 = \gamma.
\]

2 \quad \textbf{SETH implies ETH.}

Recall SETH: For every \( \varepsilon > 0 \) there is a \( k \) so that \( k\text{-SAT} \) on \( n \) variables cannot be solved in \( O(2^{(1-\varepsilon)n}) \) time.

\textit{Does SETH imply ETH?}

To address this question, let us consider a more believable hypothesis than ETH:

\textbf{More Believable ETH (MBETH):} There exists a \( k \geq 3 \) and a \( \delta > 0 \) so that \( k\text{-SAT} \) on \( n \) variables cannot be solved in \( O(2^{\delta n}) \) time.

Clearly, ETH implies MBETH. Similarly, SETH implies MBETH: SETH implies for instance that there is a \( k \) that cannot be solved in \( O(2^{0.1n}) \) time (choosing \( \varepsilon = 0.2 \)) which clearly implies MBETH.

We will actually show that MBETH is equivalent to ETH! If there is any \( k \) for which \( k\text{-SAT} \) requires exponential time, then \( 3\text{-SAT} \) also does. As a byproduct we get that SETH implies ETH.

To prove this, we use Sparsification again.

We will show that if \( 3\text{-SAT} \) is in \( 2^{\delta n} \) time for all \( \delta > 0 \), then \( k\text{-SAT} \) is also in \( 2^{\delta n} \) time for all \( \delta \).

We begin with \( F \), a \( k\)-CNF with \( n \) variables and \( m \) clauses.

Consider the standard reduction from \( k\text{-SAT} \) to \( 3\text{-SAT} \). Given a \( k\)-CNF \( F \) we perform the following conversion: for each clause \( (x_1 \lor \ldots \lor x_k) \), where the \( x_i \) are literals, replace it with \( (x_1 \lor x_2 \lor y_1) \land (\neg y_1 \lor x_3 \lor y_2) \land \ldots \land (\neg y_{k-3} \lor x_{k-1} \lor x_k) \). Each original clause gives rise to \( k - 3 \) new variables and \( k - 2 \) clauses, giving a 3-CNF with \( n + m(k - 3) \leq mk \) variables and \( \leq mk \) clauses. Now, armed with this reduction, we take a \( k\)-CNF \( F \) and first sparsify \( F \) via Theorem \[1\]. Then we apply our \( k\)-CNF to 3-CNF transformation. This yields \( 2^{\varepsilon n} \) \( k\)-CNFs with \( O(n) \) clauses, and then \( 2^{\varepsilon^2 n} \) 3-CNFs but with only \( O(nk) \) variables and clauses. We solve these CNFs with our supposedly fast algorithm for \( k\text{-SAT} \), as desired.

3 \quad \textbf{K-SUM}

\textbf{Theorem 3.1} \quad \textit{If} \( \forall \varepsilon > 0 \exists k \text{ such that } k\text{-SUM on “small” numbers is in } O(n^{\varepsilon k}) \text{ time, then ETH is false.} \)

\textbf{Proof.} \quad \textit{Let } \( F \) \text{ be a } 3\text{-CNF with } n \text{ variables and } O(n) \text{ clauses. We convert it to } F' \text{ which is a “1 in 3 SAT” instance (exactly one literal must be true in each clause). We do this by replacing each clause } (x \lor y \lor z) \text{ with } (x \lor a \lor d) \land (y \lor b \lor d) \land (a \lor b \lor c) \land (c \lor d \lor f) \land (z \lor e). \text{ This yields } 6 \text{ new variables per clause and } 5 \text{ clauses per clause, so } F' \text{ has } O(m) \text{ variables and clauses.}

\text{Partition the variables into } k \text{ groups of } n/k \text{ size each: } V_1, \ldots, V_k. \text{ Look at all possible } 2^{n/k} \text{ partial assignments for each group. We will assign a number to each partial assignment for each group, written in base } (k + 1). \text{ The number corresponding to the partial assignment } \phi \text{ has a section corresponding to its group and a section corresponding to}
clauses it satisfies. In the group section, there is a 1 for the \( V_i \) that \( \phi \) corresponds to and a 0 otherwise. The target \( t \) has a 1 for each digit in this section, forcing the numbers chosen to solve the \( k \)-SUM problem to use one assignment from each group.

For the clause section, record the number of literals for each clause \( j \) that \( \phi \) sets to true (if there is more than one for any clause, omit \( \phi \) since it satisfies too many already). The target \( t \) has a 1 for the digit corresponding to each clause \( j \).

The numbers have \( m + k \) components in base \( k + 1 \), so their size is \((k + 1)^{m+k}\) and when \( m = O(n) \) this is \( \approx k^n \) size. The number of numbers is \( N = k^{2n/k} \) so that the size of the bit representation of the numbers, \( \log(k^n) \), is \( O(k \log k \log N) \).

If \( k \)-SUM on \( n \), \((k \log k) \log n\)-bit numbers is in \( O(n^{\delta k}) \) time \( \forall \delta \), then our resulting instance can be solved in \( O((k^{2n/k})^{\delta k}) \) time, which is \( k^{O(k)}2^{\delta n} \) time. \( \square \)

4 Sparsification Revisited

We now outline how the sparsification algorithm in Theorem 1.1 works.

Suppose we have the two clauses \( c_1 = (x \lor y \lor z) \) and \( c_2 = (x \lor y' \lor z') \). If \( x \) is true, we can remove both clauses \( c_1 \) and \( c_2 \). If \( x \) is false, we can replace the two with \((y \lor z)\) and \((y' \lor z')\).

This method generalizes to finding a “weak-sunflower”, a set of clauses that share some common sub-clause (weak because the petals can intersect). Either they can all be removed and replaced with the subclause or this common subclause can be removed from each clause. There is a tradeoff between the number of clauses involved and the size of the subclause.

If a \( k \)-CNF formula \( F \) on \( n \) variables has \( > cn \) clauses, then consider the variable \( x \) that appears in the maximum number of clauses \( q \). Then the number of clauses must be at most \( 2nq \). So \( x \) must appear in \( > c/2 \) clauses. Thus any such dense enough formula must contain a weak-sunflower on \( > c/2 \) clauses with a “heart” containing a variable common to all sunflower clauses. Thus, if no large weak sunflower can be found, the formula has become sparse.

The sparsification algorithm goes through the clauses by size: iterate over \( i \) from 2 to \( k \) and then \( j \) from 1 to \( k - 1 \) (\( i \) is the size of clauses being looked at, and \( j \) corresponds to the size of the petals). Find \( \ell_i \) \( i \)-clauses that intersect in \((i - j)\) literals. In one case remove the clauses and add the core as a clause; in the other case, set all literals in the core to false (and shrink the \( \ell_i \) clauses into \( j \)-clauses).

Branch like this for depth \( \varepsilon n \), yielding \( 2^{\varepsilon n} \) leaves. Each one of these is one of the output formulas. A careful analysis states that they must be sparse.