Announcements:

- Don’t forget: ps3 due 11/18
- Project presentations will start 12/2

We will post the schedule soon

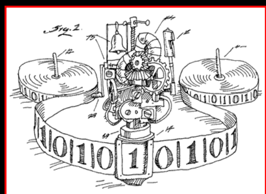
How Fine-Grained Algorithms Can Imply Circuit Complexity Lower Bounds

Ryan Williams

Circuit Complexity: A Crash Course

(Ask questions!)

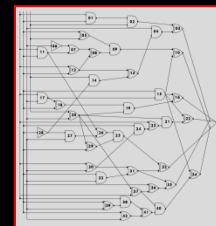
Algorithms



Can take in **arbitrarily long inputs** and still solve the problem

$f : \{0,1\}^* \to \{0,1\}$

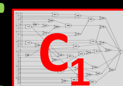
Circuits



Can only take in fixed-length inputs

$g : \{0,1\}^n \to \{0,1\}$

Circuit Family = {



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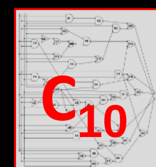

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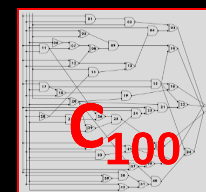

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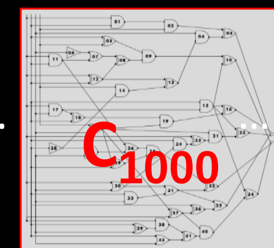


, … }

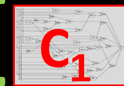
For each n, have a circuit **C_n** to be run on all inputs of length n

Circuit model has "programs with ***infinite-length descriptions***"

P/poly = { $f : \{0,1\}^* \to \{0,1\}$ computable by a **circuit family** **$\{C_n\}$** where for every n, the ***size* of C_n is at most poly(n)** }

Each circuit is "small" relative to its number of inputs

Circuit Family = {



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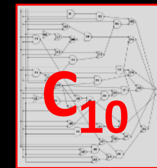

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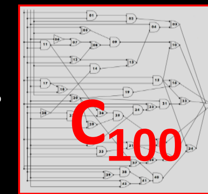

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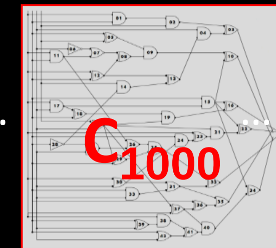

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P/poly = { $f : \{0,1\}^* \to \{0,1\}$ computable by a **circuit family** **$\{C_n\}$** where for every n, the ***size* of C_n is at most poly(n)** }

Conjecture: NP $\not\subset$ P/poly

Why study this model?

Proving limitations on what circuit families can compute is a step towards a ***non-asymptotic complexity theory:***

Concrete limitations on computing within the known universe

"Any computer solving most instances of this 10^4-bit problem needs at least 10^{125} bits to be described"

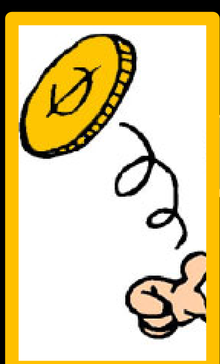
Universe stores < 10^{125} bits **[Bekenstein '70s]** **[Meyer-Stockmeyer '70s]**

Functions with High Circuit Complexity

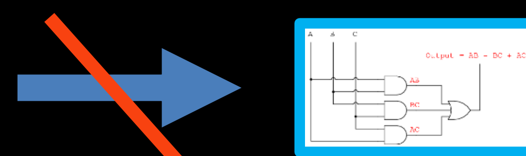
“Most” functions require *huge* circuits!

Theorem [Shannon ’49, Lupanov ‘58]

With high probability,
a randomly chosen function $f : \{0,1\}^n \to \{0,1\}$
does not have circuits of size less than $\mathbf{2^n/n}$
(and: every f has a circuit of size about $\mathbf{2^n/n}$)



0	1	0	1	1	0	1	0	0	1



Which “natural” functions exhibit this exponential behavior?

Circuits and Derandomization

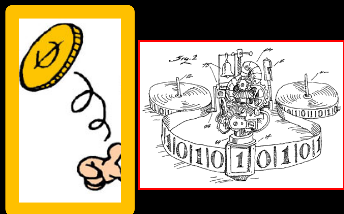
Thm [Nisan-Wigderson, Impagliazzo-Wigderson 90s]

If there is a $f : \{0,1\}^* \to \{0,1\}$
computable in $2^{O(n)}$ **time**
that does not have circuits of size at most $2^{n/100}$

Then Randomized Time $\equiv$ Deterministic Time

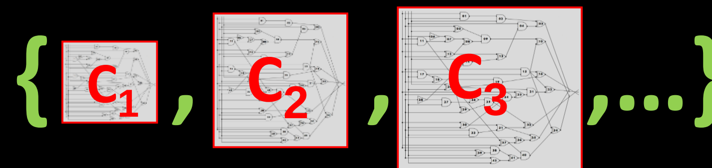
Rough intuition:
If f "looks random" to all circuits,
then f can be used to replace true randomness
in any computation!

Algorithms vs Circuit Families

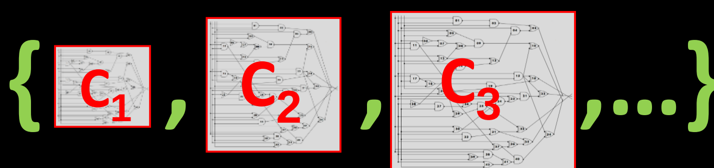


T(n) *time*
on inputs of length n

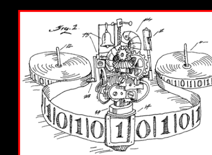
BPP is in P/poly



C_n has ≈ T(n) *size*

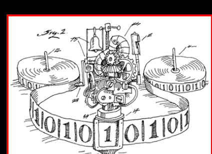


There is a family where every C_n has ≈ *size n*



No algorithm at all!

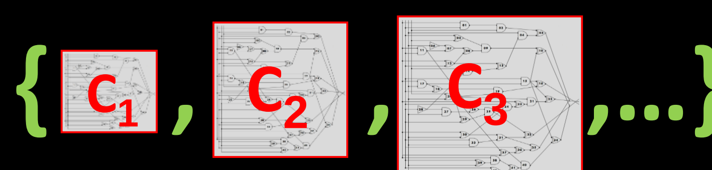
Some undecidable problems are in P/poly



EXPONENTIAL TIME
(2^n *steps*)

????

EXP is in P/poly is open!



every C_n has ≈ *n^2 size !!*

Conjecture: NP $\not\subset$ P/poly

Here endeth the Crash Course

Two Difficult Areas of Research

Fine-Grained Improvements for Solving NP Problems

Given: **Verifier V(x, y)** which reads w(|x|) bits of witness y, runs in t(|x|) time.

Find: **Deterministic or Randomized Algorithm** which:

1. Runs in ***less than*** **$2^{w(|x|)}$ t(|x|)** time
2. Given any input x, finds a witness y such that V(x,y) accepts (or conclude none)

Circuit Complexity (Non-Uniform Algorithms)

Given: Any **NP** problem Π
(or **NEXP** problem!)

Find: Sequence of algorithms **$\{A_n\}$** such that for some k:

1. **$|A_n| \leq n^k + k$**
2. On all inputs x of length **n**, **$A_n(x)$** correctly solves Π on x in **$O(n^k)$** time.

Prove that no such sequences of algorithms exist for Π

One Seems Easier Than The Other...

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Fine-Grained Improvements

- **3-SAT:** **$O(1.308^n)$ time**
- **k-SAT:** **$O(2^{n - n/k})$**
- **Hamiltonian Path:** **$O(1.66^n)$**
- **Vertex Cover:** **$O(1.3^n)$**
  on degree-3 graphs: **$O(1.09^n)$**
- **Max-2-SAT:** **$O(1.8^n)$**
- **3-Coloring:** **$O(1.33^n)$**
- **k-Coloring:** **$O(2^n)$**

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Circuit Complexity

- For all these algorithms on the LHS, we don't know how to get non-uniform algorithms (circuits) that are any better

- Best lower bound known: There is a function in NP that requires circuits of size $5n + o(n)$

- Cannot yet rule out that **NEXP** is in P/poly...

Faster Algorithms $\Longrightarrow$ Lower Bounds

Faster "Algorithms for Circuits"

[R.W. '10,'11]

Deterministic **algorithms for:**

- **Circuit SAT** in $O(2^n/n^{10})$ time with n inputs and n^k gates
- **Formula SAT** in $O(2^n/n^{10})$
- ***$\mathcal{C}$*-SAT** in $O(2^n/n^{10})$
- Given a circuit of n^k size that's either ***UNSAT***, or has $\geq$ ***2^{n-1} SAT assignments***, determine which in $O(2^n/n^{10})$ time

(Easily solved w/ randomness!)

No "Circuits for NEXP"

Would imply:

- **NEXP $\not\subset$ P/poly**
- **NEXP $\not\subset$ Poly-size Formulas**
- **NEXP $\not\subset$ *poly-size $\mathcal{C}$***

NEXP $\not\subset$ P/poly

Even Faster $\Longrightarrow$ "Easier" Functions

Better "Algorithms for Circuits"

[Murray-W. '18]

Det. algorithm for some $\epsilon > 0$:

- **Circuit SAT** in $O(2^{n-n^{\epsilon}})$ time with n inputs and $2^{n^{\epsilon}}$ gates
- **Formula SAT** in $O(2^{n-n^{\epsilon}})$
- $\boldsymbol{C}$**-SAT** in $O(2^{n-n^{\epsilon}})$
- Given a circuit of $2^{n^{\epsilon}}$ size that's either ***UNSAT***, or has $\geq$ ***2^{n-1} SAT assignments***, determine which in $O(2^{n-n^{\epsilon}})$ time

(Easily solved w/ randomness!)

No "Circuits for Quasi-NP"

Would imply:

- **NTIME[$n^{polylog\ n}$] $\not\subset$ P/poly**
- **NTIME[$n^{polylog\ n}$] $\not\subset$ NC1**
- **NTIME[$n^{polylog\ n}$] $\not\subset$ $\boldsymbol{C}$**

NTIME[$n^{polylog\ n}$] $\not\subset$ P/poly

Even Faster $\Longrightarrow$ "Easier" Functions

Fine-Grained SAT Algorithms
[Murray-W. '18]

Det. algorithm for some $\epsilon > 0$:

- **Circuit SAT** in $O(2^{(1-\epsilon)n})$ time on n inputs and $2^{\epsilon n}$ gates
- **Formula SAT** in $O(2^{(1-\epsilon)n})$
- $\mathcal{C}$**-SAT** in $O(2^{(1-\epsilon)n})$
- Given a circuit of $2^{\epsilon n}$ size that's either ***UNSAT***, or has $\geq$ ***2^{n-1} SAT assignments***, determine which in $O(2^{(1-\epsilon)n})$ time

(Easily solved w/ randomness!)

No "Circuits for NP"

Would imply:

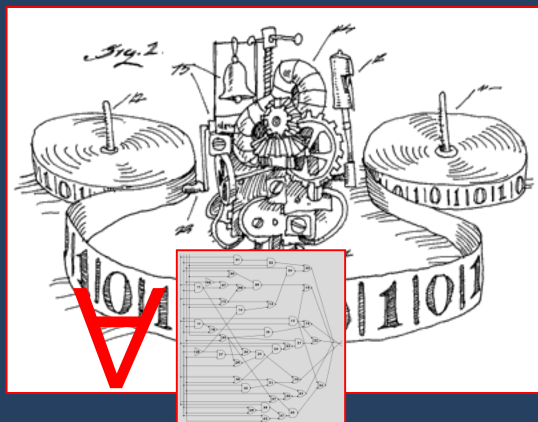
- **NP $\not\subset$ SIZE(n^k) for all k**
- **NP $\not\subset$ Formula-SIZE(n^k)**
- **NP $\not\subset$ $\mathcal{C}$-SIZE(n^k) for all k**

NP $\not\subset$ SIZE(n^k) for all k

Why on Earth would it be true?

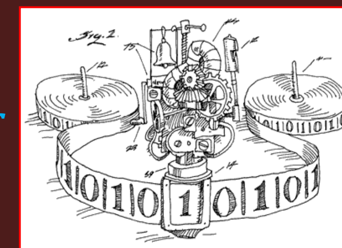
SAT? YES/NO

$\exists$



"Non-Trivial" Circuit Analysis Algorithm

$\exists$ "interesting" f



$\forall$

Circuit Lower Bounds

Concrete Lower Bounds From Algs!

Thm [R.W.'11]: NEXP $\not\subset$ ACC^0

Thm [Murray-W'18]: NTIME[$n^{poly(\log n)}$] $\not\subset$ ACC^0

NEXP = NTIME[$2^{n^{O(1)}}$]

ACC^0: ~~polynomial size, constant depth circuits with AND, OR, and MOD[m] gates for some constant m.~~

A simple but Annoying Circuit Class to prove lower bounds for (proposed in 1986 by Barrington)

How It Was Proved

Let $\mathbb{C}$ be a "typical" circuit class (like ACC^0)

Thm A [W'11]:

If for all k, $\mathbb{C}$-SAT on n^k-size is in $O(2^n/n^k)$ time, then NEXP does not have poly-size $\mathbb{C}$-circuits.

Thm B [W'11]:

$\exists\ \varepsilon$, ACC^0-SAT on $2^{n^{\varepsilon}}$ size is in $O(2^{n-n^{\varepsilon}})$ time.

An inefficiency!

Theorem B gives a much stronger algorithm than is needed in Theorem A.

This is exactly the starting point of [Murray-W'18]…

More on Theorem A

Let $\mathbb{C}$ be some circuit class (like ACC^0)

Thm A [W'11]:

If for all k, $\mathbb{C}$-SAT on n^k-size is in $O(2^n/n^k)$ time,
then NEXP does not have poly-size $\mathbb{C}$-circuits.

Idea. **Show that if we assume both:**

(1) NEXP has poly-size $\mathbb{C}$-circuits,

and

(2) a faster $\mathbb{C}$-SAT algorithm

Then we can show NTIME$[2^n] \subseteq$ NTIME$[o(2^n)]$

Proof Sketch of Theorem A

Idea. **Assume**

(1) NEXP has poly-size $\mathbb{C}$-circuits, and

(2) a faster $\mathbb{C}$-SAT algorithm

Show that NTIME$[2^n]$ $\subseteq$ NTIME$[o(2^n)]$

Take any problem L in **nondeterministic 2^n time**.
Given an input x, we "compute" L on x by:

1. Guessing some witness y of O(2^n) length.
2. Checking y is a witness for x in O(2^n) time.

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Guessing Short Witnesses

1. Guess a witness y of o(2^n) length.

Easy Witness Lemma [IKW'02]:
If NEXP has polynomial-size circuits,
then all NEXP problems have "easy witnesses"

Def. An NEXP problem L has easy witnesses if
∀ Verifiers V for L and $\forall x \in L$,
∃ poly($|x|$)-size circuit D_x such that V(x,y) accepts,
where **y = Truth Table of circuit D_x.**

1'. Guess poly($|x|$)-size circuit D_x

Verifying Short Witnesses

2. Check y is a witness for x in $o(2^n)$ time.

Assuming NEXP has polynomial-size circuits,
"easy witnesses" exist for ***every*** verifier V.
We choose a V for an NEXP-complete L so that

Checking a witness for x

$\equiv$

Solving a $\mathbb{C}$-UNSAT instance with $poly(|x|)$ size
and $n = |x| + O(\log|x|)$ inputs

Then, $2^n/n^k$ time for $\mathbb{C}$-UNSAT ➔ $o\left(2^{|x|}\right)$ time

Verifying Short Witnesses

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Checking a witness for x

$\equiv$

Distinguishing ***unsatisfiable*** circuits from those with ***many*** satisfying assignments
(Uses the PCP Theorem!)

Proof Sketch of Theorem A

Idea. **Assume**

(1) NEXP has poly-size $\mathbb{C}$-circuits, and

(2) a faster $\mathbb{C}$-SAT algorithm

Show that NTIME$[2^n]$ $\subseteq$ NTIME[o(2^n)]

Take any problem L in **nondeterministic 2^n time**.
Given an input x, we **will** "compute" L on x by:

1. **Guessing a circuit D_x of poly(|x|) size**
2. **Checking D_x encodes a witness for x in o(2^n) time**

End