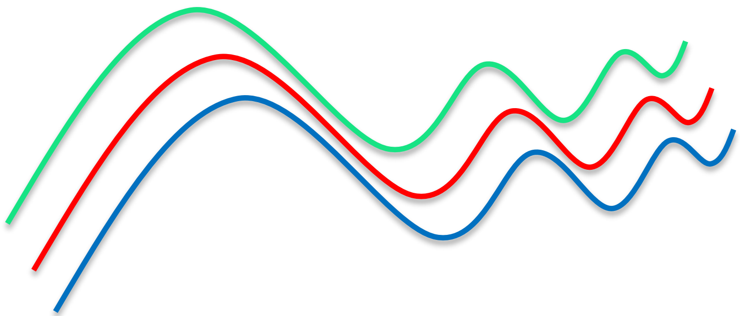
Lecture 5: Hardness for Sequence Problems under SETH and OVC



Thanks to Piotr Indyk and Arturs Backurs for some slides

Plan

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- Define sequence problems:
  - (Discrete) Frechet Distance
  - Edit Distance and LCS
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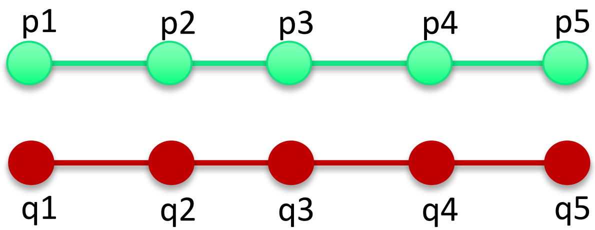
- Birds eye view on the upper bounds
  - Dynamic programming, quadratic time
- Show conditional quadratic lower bounds
  - Assuming SETH / OV, example: LCS

Walks on sequences



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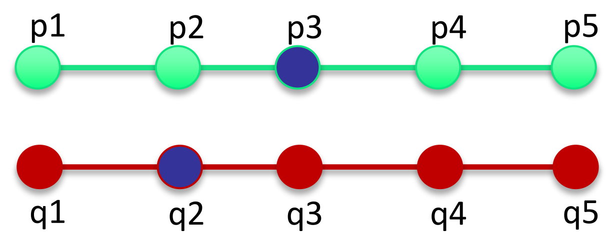

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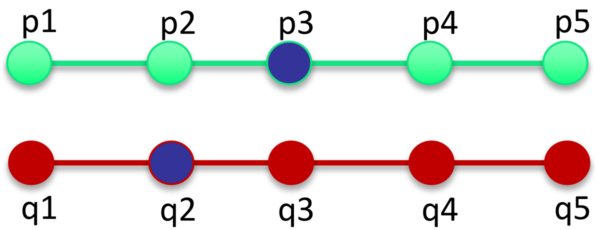

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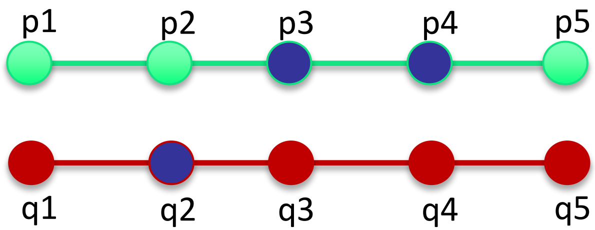

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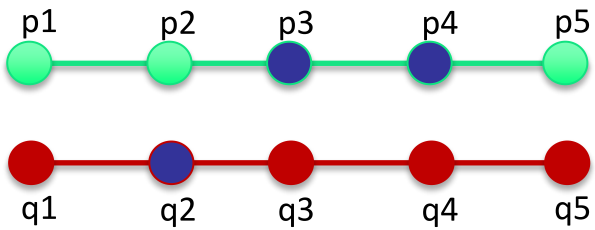

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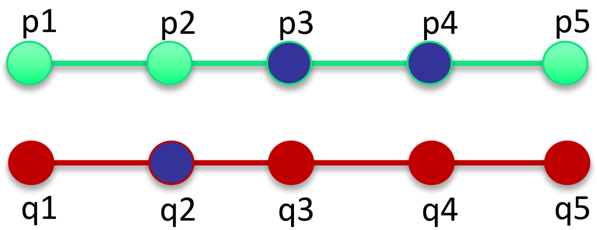

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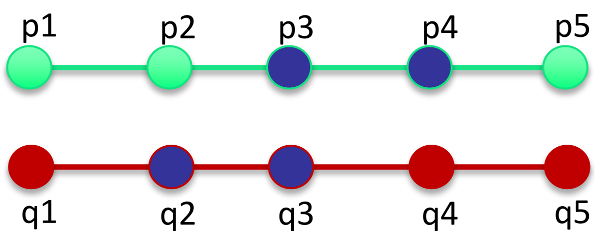

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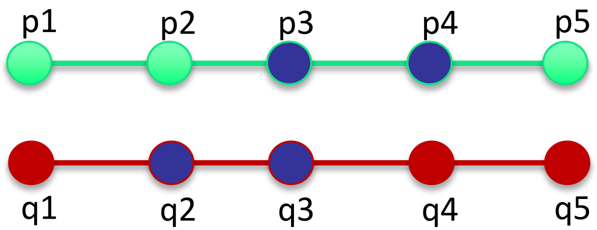

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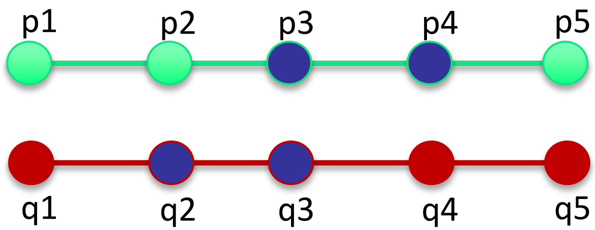

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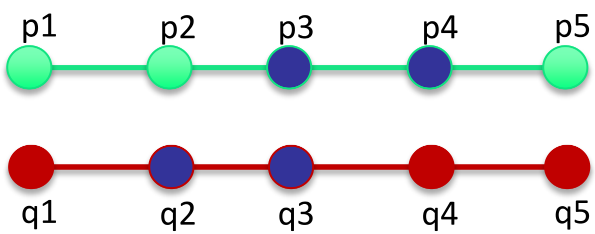

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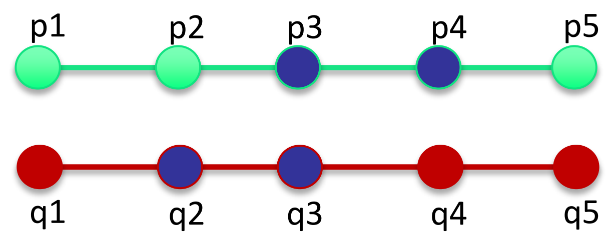

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Sequence walk problems optimize, over all such walks, some measure depending on the distances between p_i and q_j over all steps (p_i,q_j) of the walk.

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- ``Dog walking distance''
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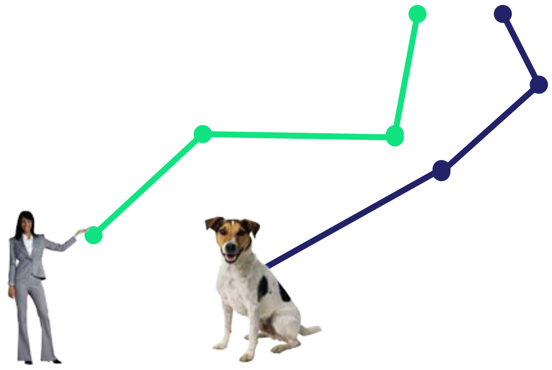

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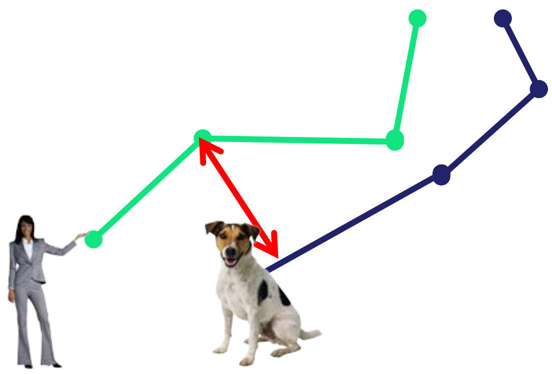

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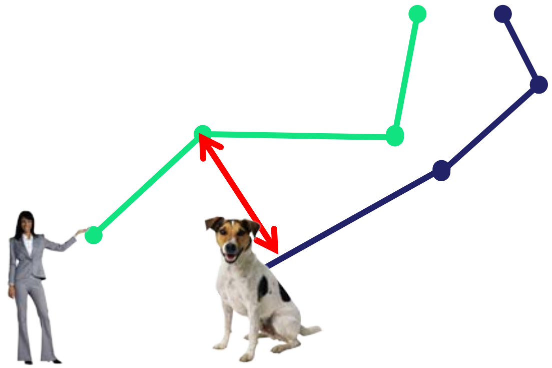

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  - Let F = set of monotone functions [0,1]→[0,1]
  - For two curves P,Q: [0,1] →R^2 :

$$D_{Fr}(P,Q) = \min_{f,g \in F} \max_{t \in [0,1]} ||P(f(t)) - Q(g(t))||$$

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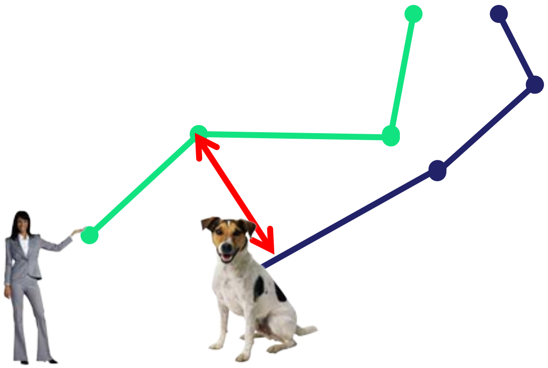

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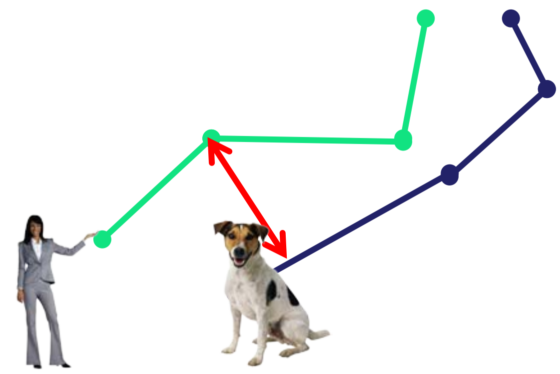

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- Many algorithms for special cases and variants

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- A simple $O(n^2)$ time dynamic programming algorithm

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Best algorithm: $O(n^2/\log n)$ [Masek-Paterson'80]

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- Approximation algorithms: $O(1)$ –approx in $O(n^{2-\varepsilon})$ time [Chakraborty-Das-Goldenberg-Koucky-Saks'18],

  $O(f(\varepsilon))$ –approx in $O(n^{1+\varepsilon})$ time [Andoni-Nowatzki'20]

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  - hard under OVH and SETH ?

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  - Dynamic Time Warping (DTW)
- Birds eye view on the upper bounds
  - Dynamic programming, quadratic time
- Show conditional quadratic lower bounds
  - Assuming SETH / OVH
  - Basic approach
  - Hardness for LCS

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  - Can be solved trivially in $O(n^2 d)$ time
  - Best known algorithm runs in $n^{2-1/O(\log c(n))}$ time, where $d=c(n)\cdot\log n$ [Abboud-Williams-Yu'15]

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- OV Hypothesis (implied by SETH):

  OV can't be solved in $n^{2-\varepsilon}\cdot d^{O(1)}$ time for any $\varepsilon > 0$.

Quadratic hardness under OVC

Theorem*: No $n^{2-\Omega(1)}$ time algorithm for EDIT, DTW, Frechet distances or LCS unless OVC fails [Bringmann'14; Backurs-Indyk'15; Abboud-Backurs-VW'15; Bringmann-Kunnemann'15]

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  - $S \subseteq \{0,1\}^d \rightarrow$ sequence x, $|x| \leq n \cdot d^{O(1)}$
  - $S \subseteq \{0,1\}^d \rightarrow$ sequence y, $|y| \leq n \cdot d^{O(1)}$

*See also [Abboud-V. Williams-Weimann'14]

Quadratic hardness under OVC

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  - distance(x,y)=small if exists $a, b \in S$ with $\Sigma_i a_i b_i = 0$
  - distance(x,y)=large, otherwise
  - The construction time is $n \cdot d^{O(1)}$
  - Gadgets for coordinates and vectors

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Next: hardness for LCS

*See also [Abboud-V. Williams-Weimann'14]

Hardness for LCS

I will present the ideas behind the proof from

[Abboud-Backurs-VW’15].

Full construction. NO full proof.

[Bringmann-Kunnemann’15] obtained an independent proof.

OV to LCS

Given vectors $\{s_1, \ldots, s_n\}, s_i \in \{0,1\}^d \ \forall i$, OV is

$$\bigvee_{i,j \in [n]} \bigwedge_{k \in [d]} (\neg\, s_i[k] \vee \neg s_j[k]).$$

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Coordinate gadgets c, e taking bits to short sequences s.t.
$LCS\big(c(x), e(y)\big) = 0$ if $x = y = 1$,
$LCS\big(c(x), e(y)\big) = 1$ if $x \cdot y = 0$.

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Vector gadgets f, g taking bit vectors to short sequences s.t. for some T
$LCS\left(f(s_i), g(s_j)\right) = T + 1$ if $s_i \cdot s_j = 0$,
$LCS\left(f(s_i), g(s_j)\right) = T$ if $s_i \cdot s_j \neq 0$.

Coordinate gadgets c, e taking bits to short sequences s.t.
$LCS(c(x), e(y)) = 0$ if $x = y = 1$,
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OV to LCS

Given vectors $\{s_1, \ldots, s_n\}, s_i \in \{0,1\}^d\ \forall i$, OV is

$$\bigvee_{i,j\in[n]} \bigwedge_{k\in[d]} (\neg\, s_i[k] \vee \neg s_j[k]).$$

Outer OR gadgets x, y taking sets of bit vectors $\{s_1, \ldots, s_n\}$, to short sequences s.t. for some Q
$LCS(x,y) = Q$ if $\forall i,j\colon\ s_i \cdot s_j \neq 0$,
$LCS(x,y) \geq Q + 1$ if $\exists i,j\colon\ s_i \cdot s_j = 0$.

Vector gadgets f, g taking bit vectors to short sequences s.t. for some T
$LCS\left(f(s_i), g(s_j)\right) = T + 1$ if $s_i \cdot s_j = 0$,
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$$\bigvee_{i,j\in[n]} \bigwedge_{c\in[d]}(\neg\, s_i[c] \vee \neg s_j[c])$$

Encoding the outer Boolean OR for OV to LCS

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Encoding the outer Boolean OR for OV to LCS

- Let S = $\{s_1, s_2, \ldots, s_n\}$ be the vectors from OV instance
- *Suppose* we have $s_i \longrightarrow$ gadget sequences $f(s_i)$ and $g(s_i)$
  LCS($f(s_i)$,$g(s_j)$) = β if $s_i \cdot s_j \neq 0$, LCS($f(s_i)$,$g(s_j)$) = $\beta + 1$ otherwise.

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Want to create sequences x and y so that **LCS(x,y) is Large** if there is an **OV pair** and **LCS(x,y) is Small** otherwise.

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Idea: *Imagine gadgets are letters.*

If no OV, LCS length is $n\ \beta$; If $s_i \cdot s_j = 0$ can align $f(s_i)$ and $g(s_j)$ and all other $f(s_k)$ with $g(s_0)$ to get LCS length $\geq (n-1)\ \beta + (\beta+1) > n\ \beta$.

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Let $S = \{s_1, s_2, \ldots, s_n\}$ be the vectors

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0 and 1 don't appear in the f and g gadgets

Attempt 2:

$Q = 0^q$, $R = 1^q$

$x = Q\, f(s_1) R\, Q\, f(s_2) R \ldots Q\, f(s_n)\, R$

$y = (Qg(s_0)\, R)^{n-1}\, Qg(s_1) R\, Q\, g(s_2)\, R \ldots Q\, g(s_n)\, R\, (Qg(s_0)\, R)^{n-1}$

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Lemma: If a 0 (or 1) is matched, its entire 0^q (or 1^q) block is matched.

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Idea: Pick q big so all Qs and Rs of x must be matched in an LCS.

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Now no $g(s_k)$ is aligned with two different $f(s_i)$ and $f(s_j)$.

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Lemma: If a 0 (or 1) is matched, its entire 0^q (or 1^q) block is matched.

Idea: Pick q big so all Qs and Rs of x must be matched in an LCS.

Now no $g(s_k)$ is aligned with two different $f(s_i)$ and $f(s_j)$.

Problem: LCS might align $f(s_i)$ with **several** $g(s_k)$.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be the vectors
Each $s_i \longrightarrow$ gadget sequences $f(s_i)$ and $g(s_i)$
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Now no $g(s_k)$ is aligned with two different $f(s_i)$ and $f(s_j)$.

Problem: LCS might align $f(s_i)$ with **several** $g(s_k)$.

The $g(s_k)$ are partitioned into blocks aligned with at most a single $f(s_i)$.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be the vectors
Each $s_i \longrightarrow$ sequences $f(s_i)$ and $g(s_i)$
$LCS(f(s_i), g(s_j)) = \beta$ if $s_i \cdot s_j \neq 0$, $\geq \beta + 1$ otherwise.
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LCS hardness idea

$Q = 0^q, R = 1^q, P = 2^r$

Attempt 3:

$x = P^{|y|} Q\ f(s_1) R\ Q\ f(s_2) R\ Q \ldots RQ\ f(s_n)\ R\ P^{|y|}$

$y = P\ (Qg(s_0)\ RP)^{n-1}\ Q\ g(s_1)\ R\ P\ \ Q\ g(s_2)\ R\ P \ldots\ \ Q\ g(s_n)\ R\ P\ (Q\ g(s_0)\ RP)^{n-1}$

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Idea:

Let $S = \{s_1, s_2, \ldots, s_n\}$ be the vectors
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Idea:

$P = 2^r$, r big but **r<<q**, so that in an LCS all Qs and Rs of x are still aligned, and also as many Ps as possible from y are aligned.

LCS hardness idea

Let $S = \{s_1,s_2,\ldots, s_n\}$ be the vectors
Each $s_i \longrightarrow$ sequences $f(s_i)$ and $g(s_i)$
$LCS(f(s_i),g(s_j)) = \beta$ if $s_i \cdot s_j \neq 0$, $\geq \beta + 1$ otherwise.
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Idea:

$P = 2^r$, r big but **r<<q**, so that in an LCS all Qs and Rs of x are still aligned, and also as many Ps as possible from y are aligned.

$\geq$ n-1 Ps of y not matched in an LCS due to the matched Qs and Rs of x.

Let $S = \{s_1,s_2,\ldots, s_n\}$ be the vectors
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$Q=0^q, R=1^q, P=2^r$

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The gadgets $f(s_i)$ and $g(s_j)$ act as letters!

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LCS length:

#Ps in y is 3n-1, and n-1 are not matched, so 2n aligned.

$2n|P| + n(|Q|+|R|) + \Sigma^n_{i=1} LCS(f(s_i), g(s_j))$, $g(s_j)$ aligned with $f(s_i)$

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$= 2nr + 2qn + n\beta$ if no orthog. pair

$\geq [2nr + 2qn + n\beta] + 1$ if 9 an orthog. pair.

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Tricky proof in paper shows the following suffice:

$|Q|, |R|, |P|, |f(s_i)|, |g(s_i)| \leq$ poly(d), so that

$|x|, |y| \leq n$ poly(d).

OV to LCS

Given vectors $\{s_1, \ldots, s_n\}, s_i \in \{0,1\}^d \; \forall i$, OV is

$$\bigvee_{i,j\in[n]} \bigwedge_{k\in[d]} (\neg\, s_i[k] \vee \neg s_j[k]).$$

Outer OR gadgets x, y taking sets of bit vectors $\{s_1, \ldots, s_n\}$, to short sequences s.t. for some Q
$LCS(x, y) = Q$ if $\forall i, j\colon\ s_i \cdot s_j \neq 0$,
$LCS(x, y) \geq Q + 1$ if $\exists i, j\colon\ s_i \cdot s_j = 0$.

Vector gadgets f, g taking bit vectors to short sequences s.t. for some T
$LCS\left(f(s_i), g(s_j)\right) = T + 1$ if $s_i \cdot s_j = 0$,
$LCS\left(f(s_i), g(s_j)\right) = T$ if $s_i \cdot s_j \neq 0$.

Coordinate gadgets c, e taking bits to short sequences s.t.
$LCS\big(c(x), e(y)\big) = 0$ if $x = y = 1$,
$LCS\big(c(x), e(y)\big) = 1$ if $x \cdot y = 0$.

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$LCS(c(x), e(y)) = 0$ if $x = y = 1$,
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Done!

c(0) = 46 e(0) = 64
c(1) = 4 e(1) = 6

LCS(c(1),e(1)) = 0, and
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All that remains!

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3,5 brand new symbols
u is large,
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where $r = u(d+1)+d-1$, $u > d$.

$LCS(f(s)_i, g(s_j)) = r$ if $s_i \cdot s_j \neq 0$
and
$LCS(f(s_i), g(s_j)) = r+1$
otherwise.

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OV to LCS

Given vectors $\{s_1, \ldots, s_n\}, s_i \in \{0,1\}^d \; \forall i$, OV is

$$\bigvee_{i,j\in[n]} \bigwedge_{k\in[d]} (\neg\, s_i[k] \vee \neg s_j[k]).$$

Outer OR gadgets x, y taking sets of bit vectors $\{s_1, \ldots, s_n\}$, to short sequences s.t. for some Q
$LCS(x, y) = Q$ if $\forall i, j$: $s_i \cdot s_j \neq 0$,
$LCS(x, y) \geq Q + 1$ if $\exists i, j$: $s_i \cdot s_j = 0$.

Vector gadgets f, g taking bit vectors to short sequences s.t. for some T
$LCS\left(f(s_i), g(s_j)\right) = T + 1$ if $s_i \cdot s_j = 0$,
$LCS\left(f(s_i), g(s_j)\right) = T$ if $s_i \cdot s_j \neq 0$.

Coordinate gadgets c, e taking bits to short sequences s.t.
$LCS(c(x), e(y)) = 0$ if $x = y = 1$,
$LCS(c(x), e(y)) = 1$ if $x \cdot y = 0$.

Done!

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c(0) = 46 e(0) = 64
c(1) = 4 e(1) = 6

LCS(c(1),e(1)) = 0, and
LCS(c(x),e(y)) = 1
for (x,y) $\neq$ (1,1).

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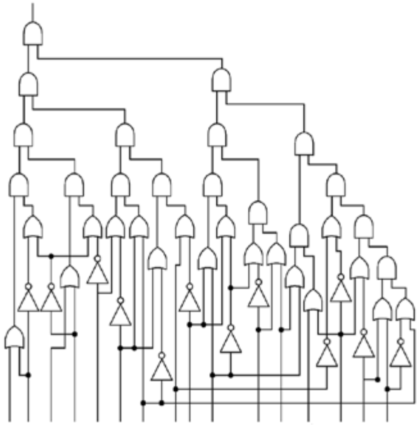
  - On much more believable assumptions!

Circuit-Strong-ETH

- SETH is ultimately about SAT of *linear size* CNF-formulas



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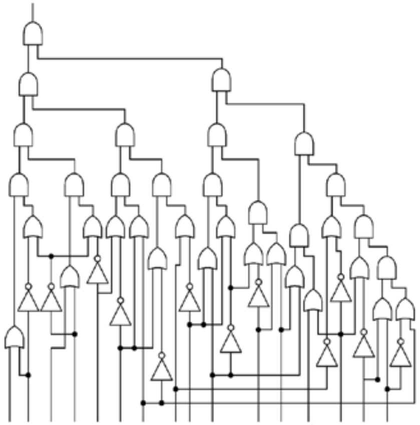

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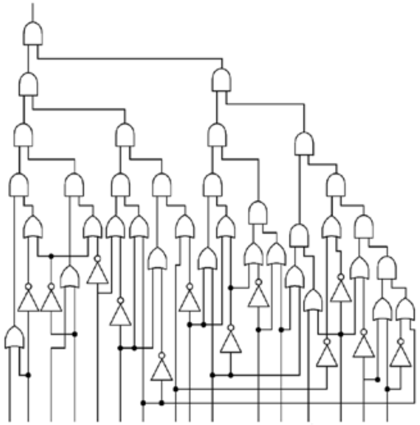

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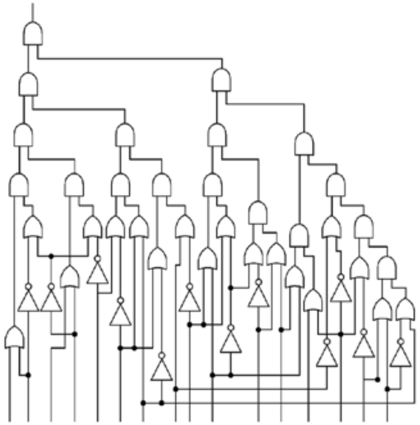

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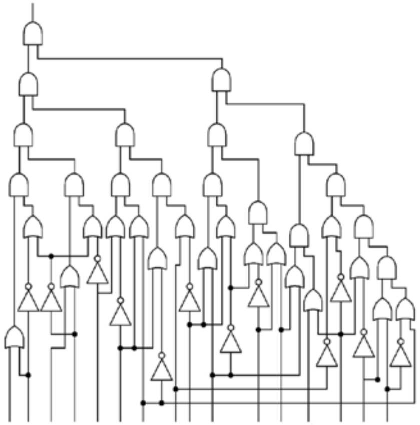

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E.g. NC-SETH should be much more believable!

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AHVW'15:
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Best alg: $\frac{n^2}{\log^2 n}$