ANOTHER WAY OF TEACHING CALCULUS: ONE-LINE DETERMINISM

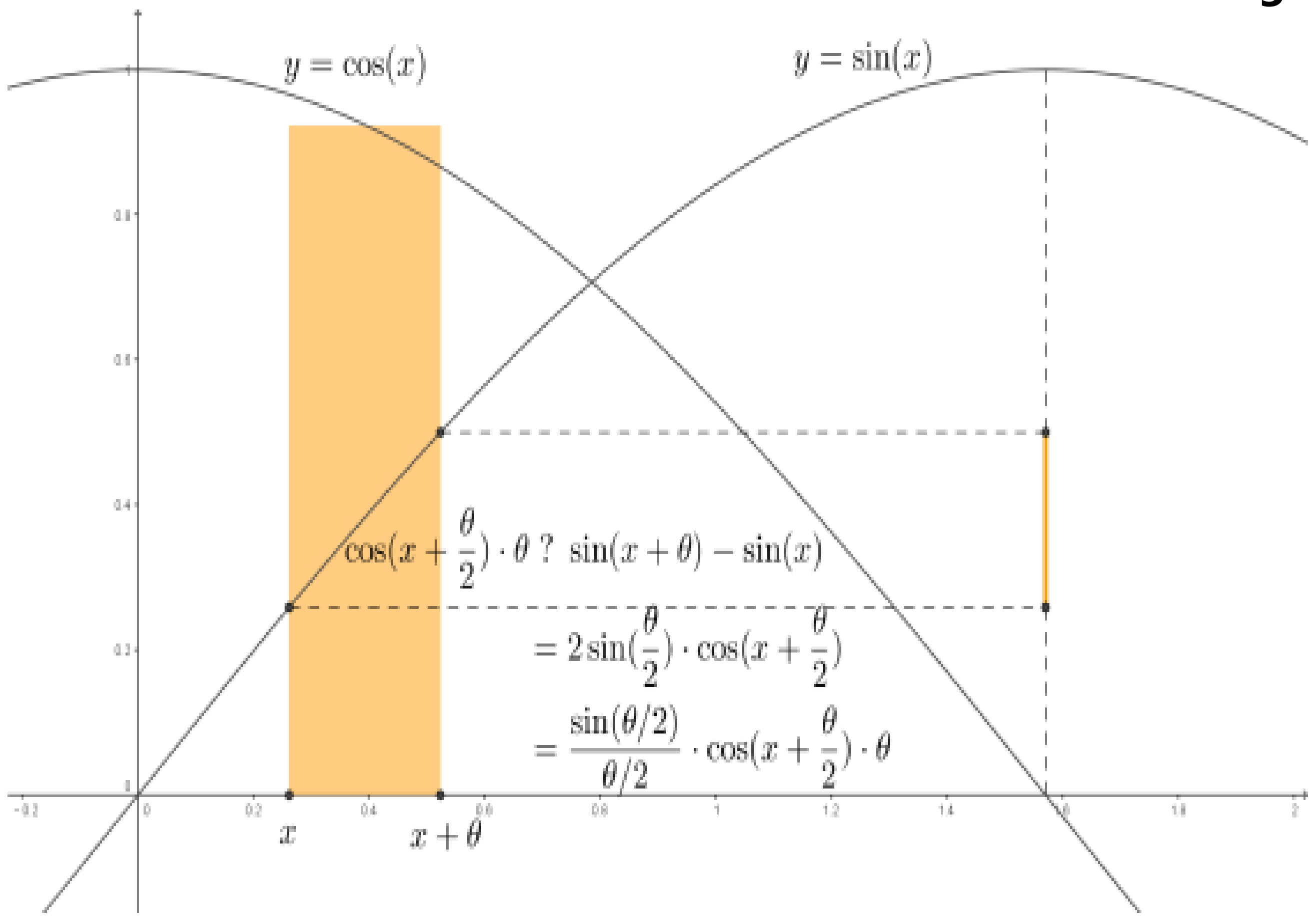
FROM SCIENCE-NET BLOG OF QUN LIN (TRANSLATED BY LIANG WANG (MIT)

http://blog.sciencenet.cn/home.php?mod=space&uid=1252&do=blog&view=me&from=space）

The complexity of calculus (including its content and method) can be exposed through middle school exercises.These exercises are determined by one-line and QED in two lines, and so as the theory (fundamental theorem) !

The spirit of our teaching is “short” that means “one exercise with one page”

Example 1 . Calculate the area under cos : the area reduce to the height



The height of a line segment under sin (from x to $x+\theta$)

$\leftrightarrow$ the area of a rectangle under cos

$$\sin(x+\theta)-\sin(x)=\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\cdot\cos(x+\frac{\theta}{2})\cdot\theta \qquad \text{(where } 1\leftarrow\cos\frac{\theta}{2}<\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}<1)$$

By summation，the total length $\leftrightarrow$ The sum of all rect areas under cos

$$\sin(b)-\sin(a)=\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\sum_{x}\cos(x+\frac{\theta}{2})\cdot\theta \quad OR \ \sum_{x}\cos(x+\frac{\theta}{2})\cdot\theta=\frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}}(\sin(b)-\sin(a))$$

$\rightarrow$ The whole area under cos (when $\theta\rightarrow 0$),denoted by

$$\int_a^b\cos(x)dx=\sin(b)-\sin(a)$$

The above two-line triangulation analysis is “determined by the first line (i.e. calculating the sub-height) and QED in two lines (i.e. calculating the total height)”, although each line needs explanation.

Summarized as a single sentence:

Area = Height (adage : pancake = loaf , jargon :2d

becomes 1d)

Exercise (similar to example 1):

$$\int_a^b -\sin(x)dx = \cos(b) - \cos(a)$$

After solving the area of cos/sin , how are the other curves ?

Example 2: Example 1 (i.e. $\cos / \sin$) is the most compact one. Let us try another example, e.g. x^3 (with "tail" ,i.e. residue)

$$(x+\theta)^3 - x^3 = 3x^2\theta + 3x\theta^2 + \theta^3 = 3(x+\frac{\theta}{2})^2\theta + \frac{1}{4}\theta^3 \qquad (1)$$

We call the residue "tail", and $tail = \frac{1}{4}\theta^3$. *Thus,*

$$b^3 - a^3 = \sum_x 3(x+\frac{\theta}{2})^2\theta + \frac{1}{4}(b-a)\theta^2 \qquad (2)$$

$$\rightarrow \text{The whole area under } 3x^2 \text{ (when } \theta \rightarrow 0\text{), denoted by } \int_a^b 3x^2dx$$

In general, we can also use the following:

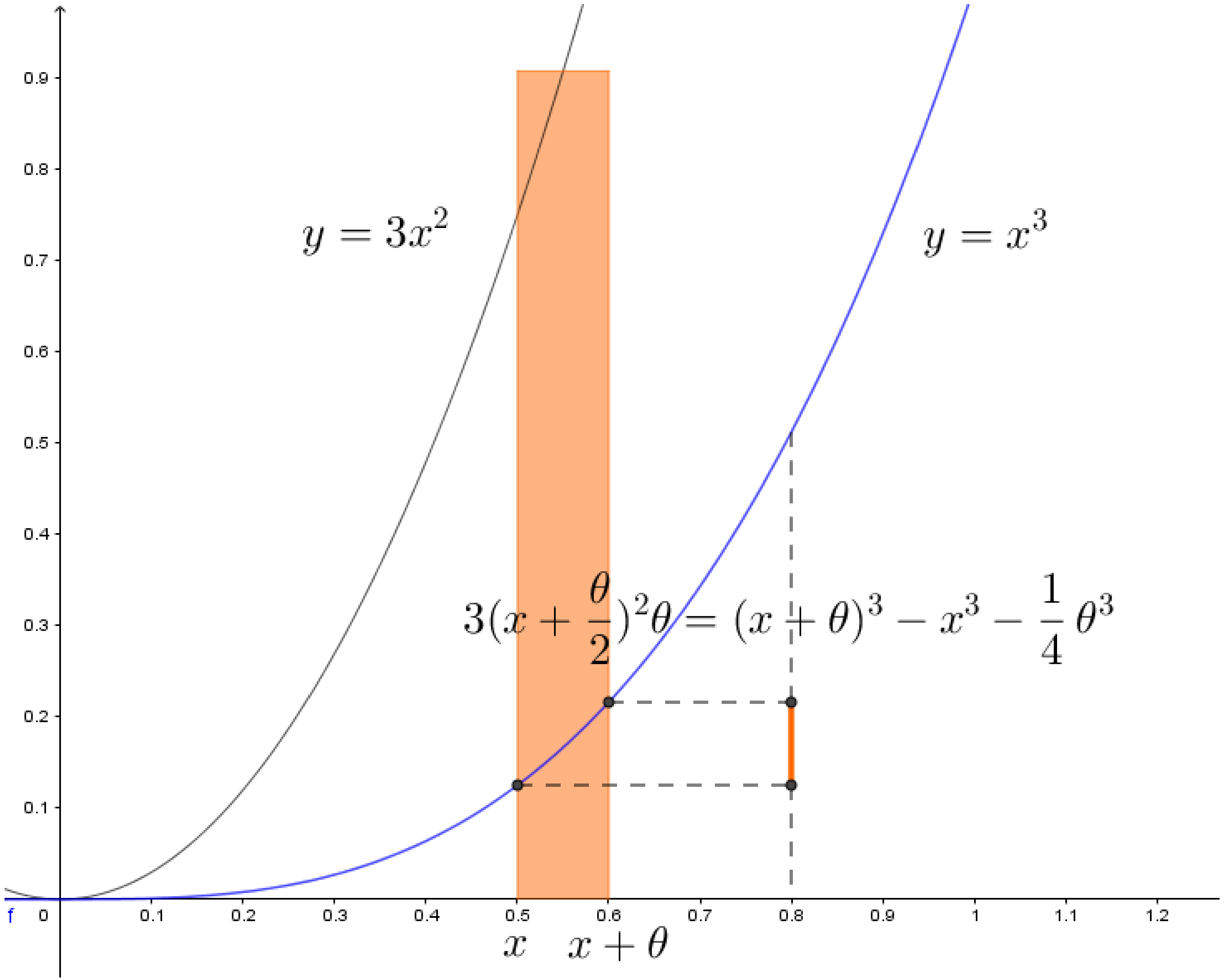
$$(x+\theta)^3 - x^3 = 3x^2\theta + tail\cdot\theta, \quad tail = 3x\theta + \theta^2 \qquad (1)$$

$$|tail| \le u.b.(\theta) = 3b\theta + \theta^2 \qquad (indep.\ of\ x)$$

$$b^3 - a^3 = \sum_x 3x^2\theta + \sum_x \text{tail}\cdot\theta, \quad \left|\sum_x tail\cdot\theta\right| \le \sum_x |tail|\cdot\theta \le u.b.(\theta)\cdot\sum\theta = (3b\theta+\theta^2)\cdot(b-a) \rightarrow 0 \qquad (2)$$

$$\Rightarrow b^3 - a^3 = \int_a^b 3x^2dx \qquad \text{where} \left|\sum_x\ \right| \le \sum_x |\ |$$

Above two-line algebra exercises: determined by (1) (i.e. calculating the sub-height) and QED in (2) (i.e. calculating the total height)



$y = 3x^2$
$y = x^3$
$3(x+\frac{\theta}{2})^2\theta = (x+\theta)^3 - x^3 - \frac{1}{4}\theta^3$
x
$x+\theta$
f
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2

Exercise：Yu Li etc. analyze the function $\sqrt{x}$

$$\sqrt{x+\theta}-\sqrt{x}=\frac{\theta}{\sqrt{x+\theta}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\theta+\left(\frac{1}{\sqrt{x+\theta}+\sqrt{x}}-\frac{1}{2\sqrt{x}}\right)\theta=\frac{1}{2\sqrt{x}}\theta+\text{tail}\cdot\theta$$

$$\left|\text{tail}\right|=\left|\frac{1}{\sqrt{x+\theta}+\sqrt{x}}-\frac{1}{2\sqrt{x}}\right|=\left|\frac{-\theta}{2\sqrt{x}(\sqrt{x}+\sqrt{x+\theta})^2}\right|\le \text{u.b.}(\theta)=\frac{\theta}{8a\sqrt{a}}\quad \text{(independent of } x\text{)} \qquad (1)$$

$$\sqrt{b}-\sqrt{a}=\sum_x\frac{1}{2\sqrt{x}}\cdot\theta+\sum_x \text{tail}\cdot\theta,\qquad \left|\sum_x \text{tail}\cdot\theta\right|\le\frac{\theta}{8a\sqrt{a}}(b-a)\to 0, \qquad (2)$$

$$\to\int_a^b\frac{1}{2\sqrt{x}}dx$$

Two-line algebra exercise again: determined by (1) (i.e. calculating the sub-height) and QED in (2) (i.e. calculating the total height)

They also analyzed other exercises, e.g. $\tan x, \ln x, e^x$
In every example, the two lines (1)(2) appear. The more examples we analyze, the more we believe in these two lines. Even from only one example, e.g. x^3**, we can also derive the general theorem. (Golden example)**

3.Generalization： Line(1)(2)in example x^3 (or $\sqrt{x}$) includes all complexity of calculus (i.e. the fundamental theorem) . The case of the general function $f(x)$ is just a copy of the case x^3 (or $\sqrt{x}$). Let $g(x)$ denote the dominant term (i.e. $3x^2$ or $\frac{1}{2\sqrt{x}}$). Then, eq. (1) is the definition (or hypothesis) that the dominant term exists ($\therefore$ no need to prove).

$$f(x+\theta)-f(x)=g(x)\theta+\mathrm{tail}(x,\theta)\cdot\theta,\quad |tail(x,\theta)|\le u.b.(\theta)\to 0 \quad (indep.of \quad x) \quad (1)$$

$$(\text{new condition} \Leftrightarrow \text{old condition}：\text{"tinny" tail} \to 0)$$

$$\Rightarrow \text{Conclusion}: f(b)-f(a)=\sum_x \mathrm{g}(x)\theta+\sum_x \mathrm{tail}(x,\theta)\cdot\theta,$$

$$\left|\sum_x \mathrm{tail}(x,\theta)\cdot\theta\right| \le \sum_x |\mathrm{tail}(x,\theta)|\cdot\theta \le \mathrm{u.b.}(\theta)\cdot\sum\theta=\mathrm{u.b.}(\theta)\cdot(b-a)\to 0 \qquad (2)$$

$$\Rightarrow Fundamental \quad Theorem: f(b)-f(a)=\int_a^b g(x)dx \qquad (3)$$

Thus， definition (1)⊃ theorem(3).That is, the one-line determinism

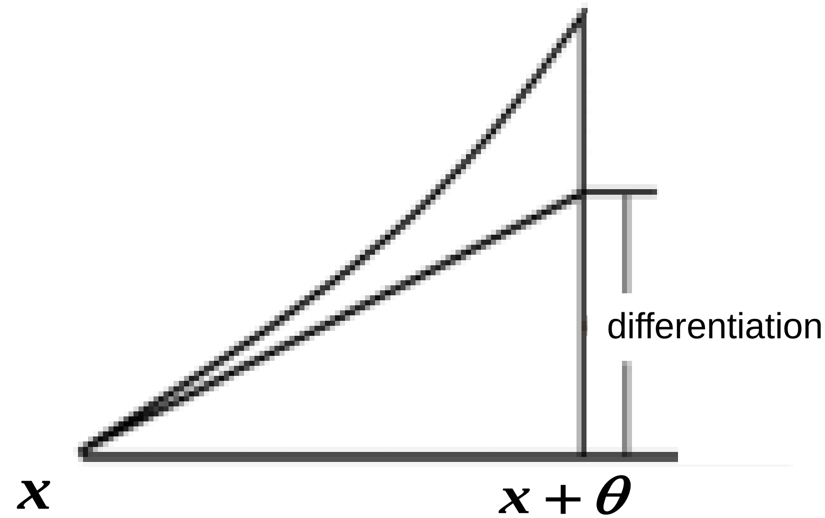
This is called “teaching by examples”. That is, introduce the theorem by exercises. The exercises are determined by one-line, so as the theorem.

4.What is the dominant term $g(x)$? Is it unique? From (1), the secant of $f(x)$ at the point x is as following:

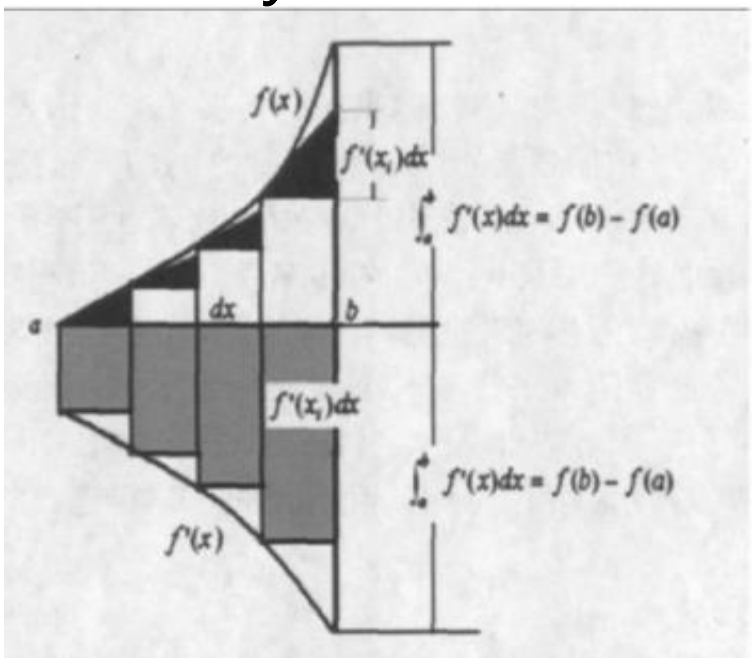
$$\frac{f(x+\theta)-f(x)}{\theta}=g(x)+tail(x,\theta)\to g(x) \quad (when \quad \theta\to 0) \quad (4)$$

The left hand side is the slop of the secant of $f(x)$ at x. The right hand side is the limit of the slop of the secant. This limit is defined as the tangent of the curve $f(x)$ at the point x (denoted by $f'(x)$.). It is unique. Thus, $f'(x)\cdot\theta$ is the sub-height under the tangent, which is also called the differentiation. The tangent is the straight line that is closest to the curve in the neighborhood around the point x.

The fundamental theorem is: the integration of the differentiation. (See the following Figure.)
The formula (4) is somehow like a "bridge" connecting the two fields about integration and differentiation



Originally,the differential $f'(x)dx$ is the sub-area under the curve $f'(x)$, but it now returns to the tangent-height of the original curve $f(x)$ at x ,and so the original sum of sub-areas returns to the sum of tangent-heights,
which is naturally close to the total height

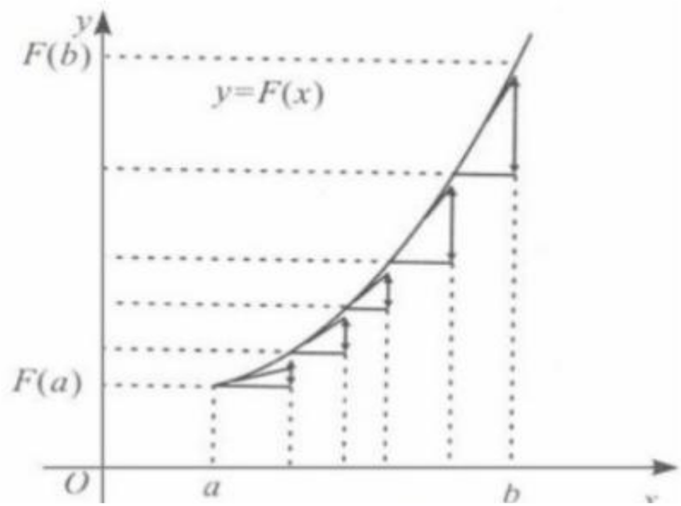


curved triangle and triangles

curved trapezoid and rectangles

The conclusion of exercise 1,hence, has a satisfactory explanation.
And, the above figure calls the liaison map of calculus.
The fundametal theorem is concluded from the previous examples, and generated through imagination, testing and conclusion.

The height-calculation figure can be thought as climbing a mountain



Qun Lin's height-calculation figure for calculus

P.235 in Jingzhong Zhang's book and P.62 in "National scientific syllabus "

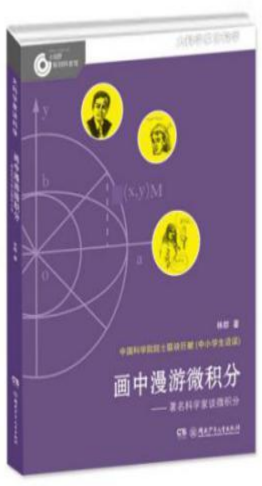
When the mountain is very high, it is difficult to calculate its height. The mountain is winding. Thus, we can only measure the sub-height of pieces, and then add up to obtain the total height. Even so, the height of each piece is still unknown. The mountain can be treated as a function whose specific form is still unknown. We make a tangent in the current position, replace the small section of the mountain by the tangent, and then calculate the height of the tangent line. Moreover, error comes from replacing curved piece by the corresponding tangent. The smaller the piece is, the better approximation becomes. Furthermore, as the segment becomes smaller and smaller, The total sub-height approaches the mountain's real height. (comments by You Chunguang and Xie Manting)

This fig first appeared in "Guangming Daily" and "People's Daily" (1997)

For most students, that is all they need to know.

The hypothesis (1) is to replace "derivative on every point" (appears in most textbooks) by "derivative on interval", which also appears in the Lax's book. These two definitions are equivalent in concept, but not equivalent in efficiency (like 2+9⇔9+2).

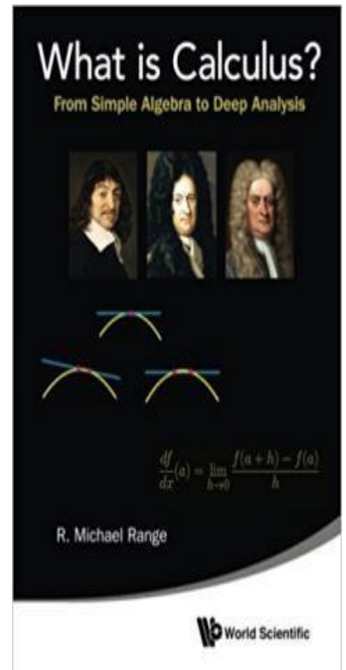
Traditional proof（with 100 lines） is reduced to only 2 lines. We kept our promise: proof with more than four lines will not be included in the textbook.（This four-line proof first appears in Qun Lin's book "Wander calculus through figures" (1988 and republished in 2017)）



Note that for the usual functions (whose second derivative is bounded), **we can suppose (1) is true, i.e.**

$$|\mathrm{tail}(x,\theta)| \le C \cdot \theta$$

Jingzhong Zhang (2012, P253), Michael Range (2016, P257)



Having the fundamental theorem we can solve more exercises:

primitive function	derivative
$\sin x$	$\cos x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arcsin x + x\sqrt{1-x^2}$	$2\sqrt{1-x^2}$
...	...
$\frac{1}{2}f^2$	$f\ f'$

Calculate the circumference and area of the unit circle (by the left table)

$$4\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 4\int_0^1 (\arcsin x)' dx = 2\pi,$$

$$4\int_0^1 \sqrt{1-x^2}dx = 4\int_0^1 \frac{1}{2}(\arcsin x + x\sqrt{1-x^2})' dx = \pi$$

Instruction:

A area (or circumference) = A height

The good way to solve the following problem

$$\int_a^b f\ f' dx = \int_a^b \frac{1}{2}(f^2)' dx = \frac{1}{2}(f^2(b) - f^2(a))$$

The bad way：using integration by parts.

Summarized by Hongtao Chen：

Why we learn mathematics? Maybe primary and secondary school students will answer: “you can calculate the circumference and area of a polygon." However, for a slightly more complex graph with the curved edges (such as the circle), they do not know how to calculate the circumference and area. Or they only remember the formula, but do not know the reason. Thus, calculus is needed, and the fundamental theorem plays the key role. This is the most useful tool of the advanced mathematics. You can master this tool by just learning the first few pages with few examples. Thus, this is the most economical and most effective way of learning calculus. This new method of teaching calculus not only greatly promotes the extension of mathematical knowledge, but also solves more practical problems, including the following two very useful exercises.

4. From the fundamental theorem, other theorems become exercises（Thus, only one theorem）

Exercise 1 The sign of derivative $\Rightarrow$ monotonicity of function (without using the mean value theorem)

$$f'(x)\begin{array}{l} >0 \\ <0 \\ =0 \end{array} \Rightarrow \begin{array}{l} f\uparrow \\ f\downarrow \\ f=c \end{array}$$

"Intro. to advanced math." (by Luogeng Hua) and Jingzhong Zhang' book break the monopoly of this approach. Instead of taking derivative, they use the elementary method to solve many optimization problems, which is called the direct approach.

Exercise 2 Fundamental theorem $\Rightarrow$ Taylor formula

Use iterated integral, rather than multiple integral. Each step is still no more than four lines：

First order： $$\int_0^s f'(x)dx = f(s) - f(0)$$

Second order： $$\int_0^{s_1}\int_0^{s_2} f''(x)dxds_2 = \int_0^{s_1}[f'(s_2) - f'(0)]ds_2 = f(s_1) - f(0) - f'(0)s_1$$

Third order:

$$\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} f'''(x)dxds_3ds_2 = \int_0^{s_1}\int_0^{s_2}[f''(s_3) - f''(0)]ds_3ds_2$$

$$= \int_0^{s_1}[f'(s_2) - f'(0) - f''(0)s_2]ds_2 \quad = f(s_1) - f(0) - f'(0)s_1 - \frac{1}{2}f''(0)s_1^2$$

Order n+1:

$$\int_0^{s_1}\cdots\int_0^{s_n}\int_0^{s_{n+1}} f^{(n+1)}(x)dxds_{n+1}\cdots ds_2 = f(s_1) - f(0) - f'(0)s_1 - \cdots - \frac{1}{n!}f^{(n)}(0)s_1^n$$

or $$f(s_1) = f(0) + f'(0)s_1 + \cdots + \frac{1}{n!}f^{(n)}(0)s_1^n + \int_0^{s_1}\cdots\int_0^{s_n}\int_0^{s_{n+1}} f^{(n+1)}(x)dxds_{n+1}\cdots ds_2$$

The last term can not be calculated, and can be ignored in general (The absolute truth is unknown or too complicated, and thus should be replaced by the relative truth (i.e. the polynomials)**) . The Taylor formula simplifies elementary functions (**complicated to calculate**) as polynomials (+ − × ÷) . The simplest cases include:**

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, x \in R \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, x \in R$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots, x \in [-1,1]$$

Especially, $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Exercise 3 differential equation: find $f(x)$ such that

$$f'(x) = g(x) \qquad f(0) = f_0$$

$$\Rightarrow \int_0^x g(t)dt = \int_0^x f'(t)dt = f(x) - f(0) \Rightarrow f(x) = f_0 + \int_0^x g(t)dt$$

(See《Differential Equation and Triangulation》2005 Tsinghua Press)

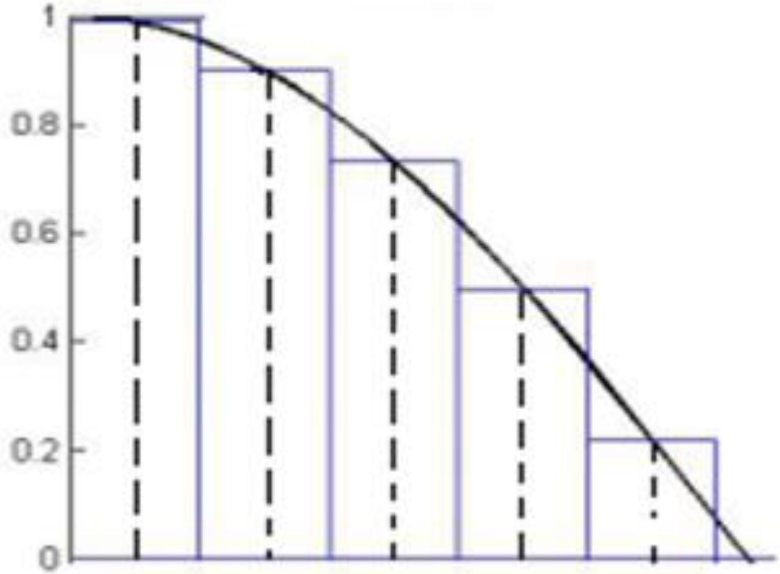
http://blog.sciencenet.cn/home.php?mod=attachment&id=65112

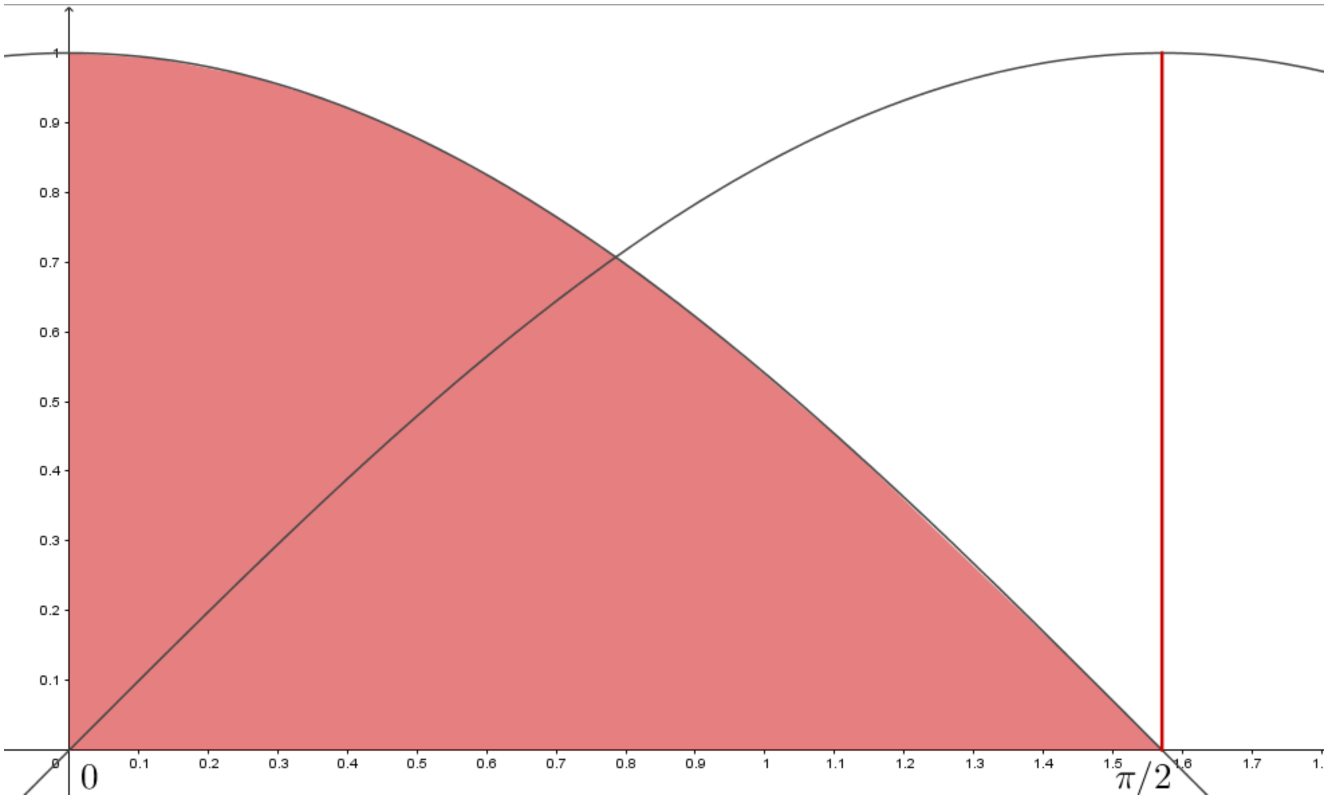
《A Great Way To Care II》

This is a competition of teaching calculus：which proof is shorter and easier to understand (with rigorous proof).
Note that this approach can be extended to the case of abstract function (which is defined on the normed space). The only change is to replace the absolute value by the norm （See Inter J Inf & Sys Sci，V.2(3)281-284，2006)

Talking to the pupils

Once I was in a primary school. The edge of the classroom is curved, while the floor is covered with rectangular covers. (It is natural to replace the curved edge by “steps”, although it’s not very accurate.)



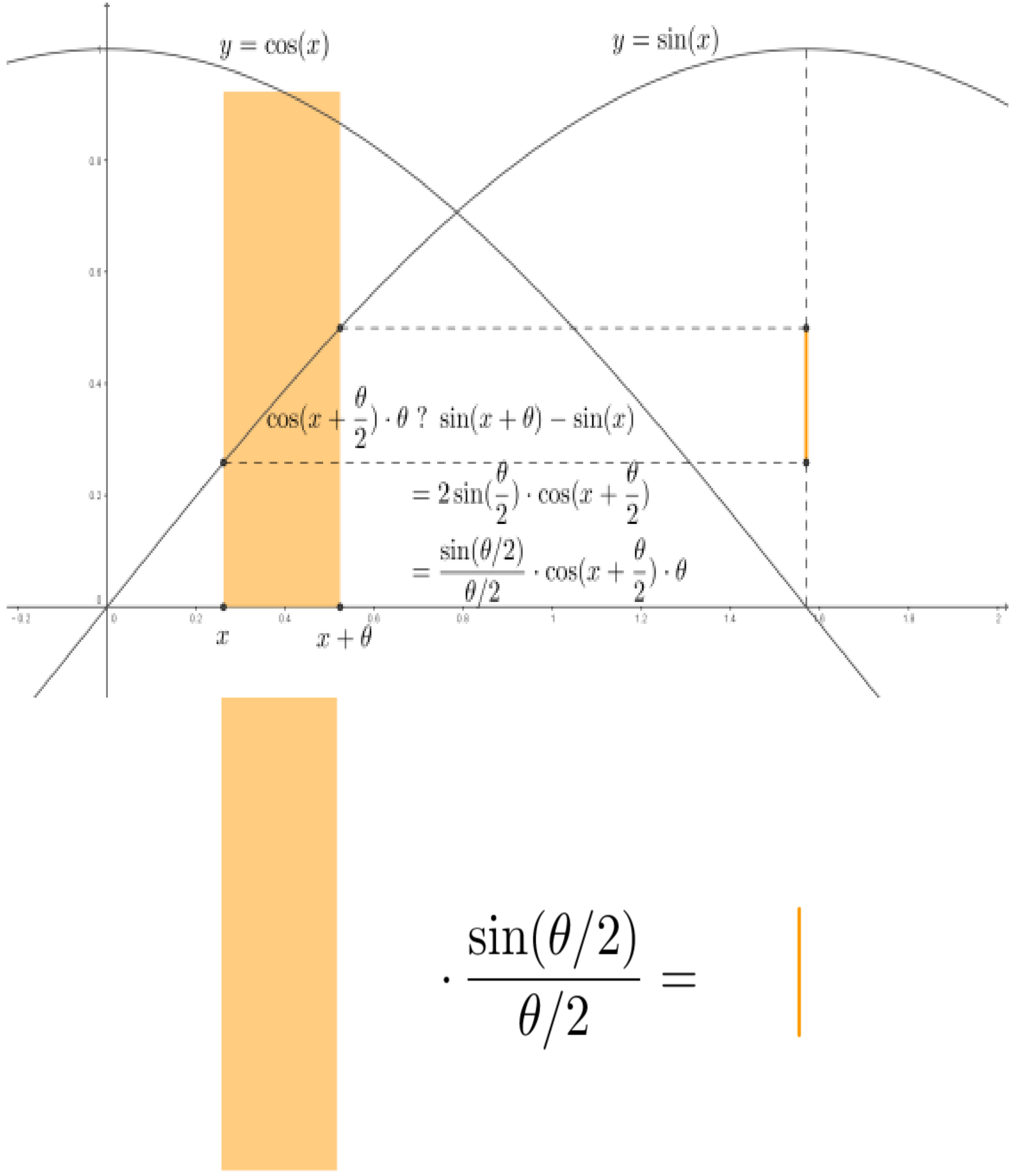


At that moment, the approach of giving the "fundamental theorem of calculus" comes into my mind , see the left Figure

Summarized as a single sentence:

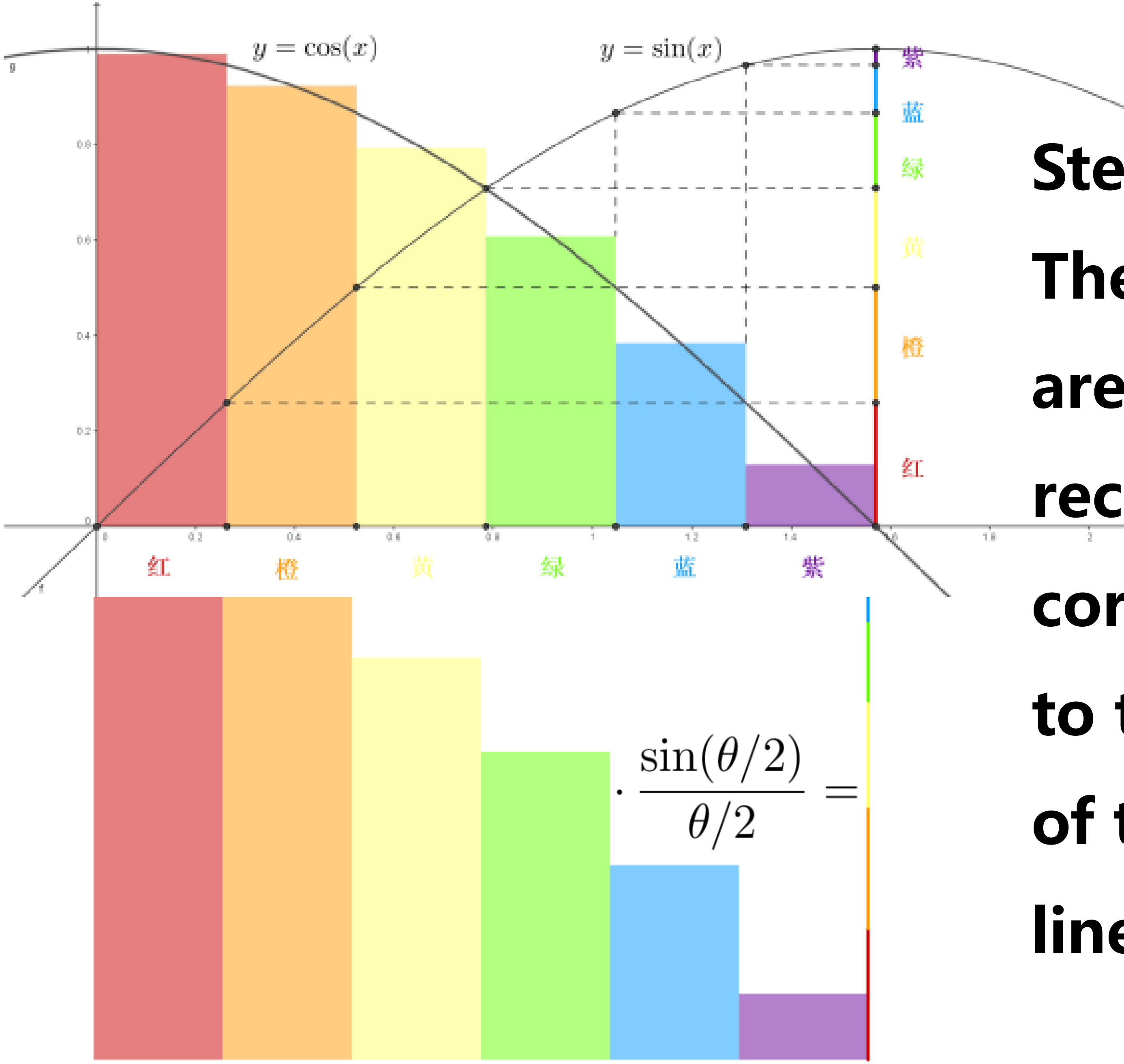
Area = Height (adage : pancake = loaf , jargon :2d becomes 1d)

Why? See the animation made by Yu Li:



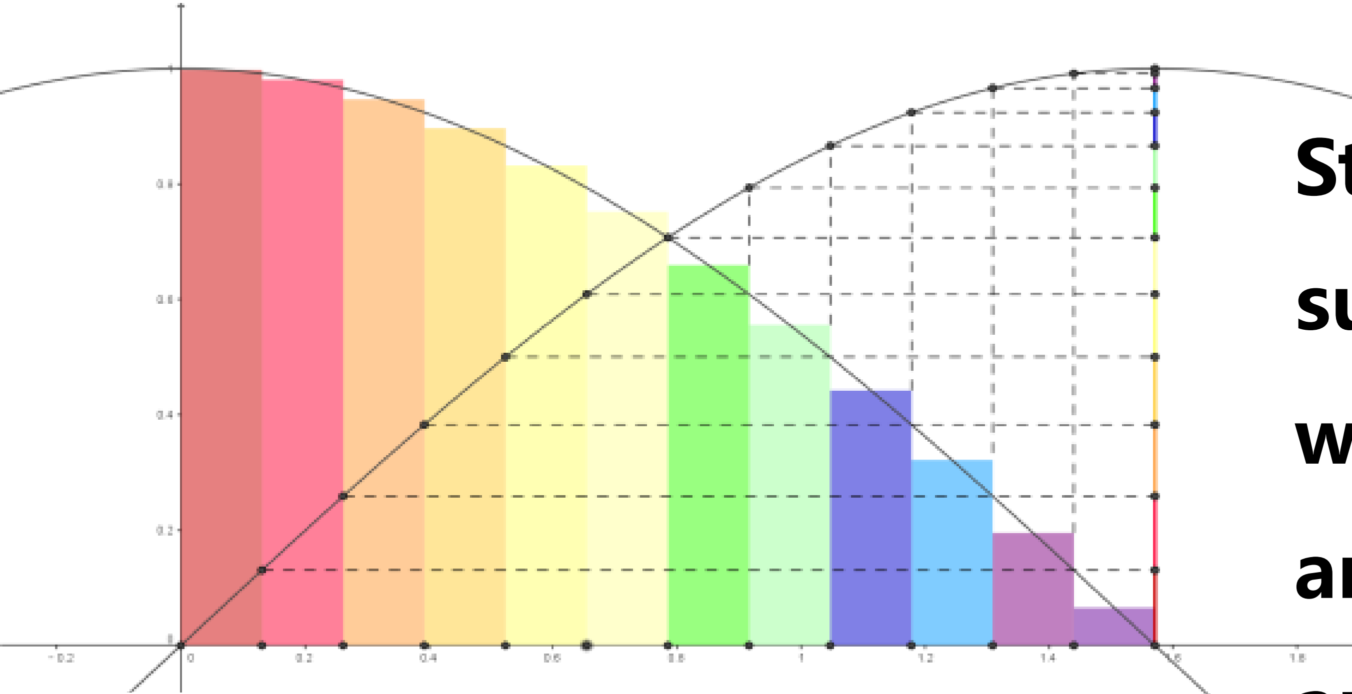
Step 1, split:

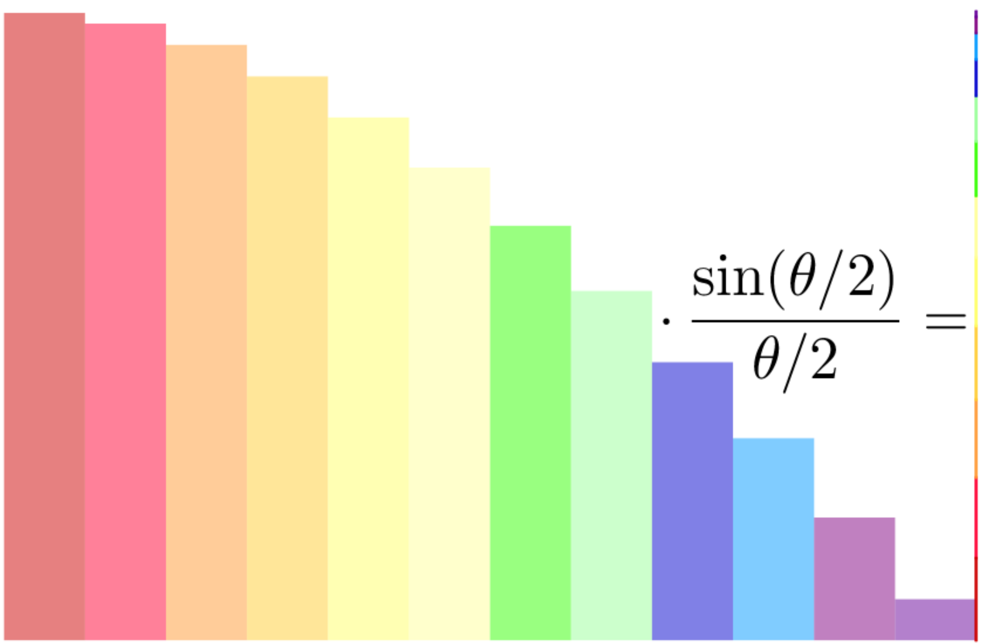
The area of a sub-rectangle Corresponds to the length of a line segment



Step 2, merge

The summed area of all the rectangles corresponds to the length of the whole line segments





$$\cdot \frac{\sin(\theta/2)}{\theta/2} =$$

Step 3, subdivide the whole space and the total area “goes to” the whole length of the line segment

Appendix

1、A young American post-doct said:
Your book opens a window for me and inspires me to think about a lot of problems from a different viewpoint. The story of "straight edge triangle and curved triangle" really tells the essence of calculus. The statement of “decomposing the big problem as small problems" is really the spirit of calculus. Looking back on our lives, this spirit is everywhere. For instance, in the field of computer science, the most classic and important algorithms is "divide and conquer”, whose basic idea is “to decompose the big problem as small problems ”. The most famous example might be the fast Fourier transform (FFT) . Your book lets me know that I still don’t quite understand calculus now (even if after I have learned calculus several times). As you mentioned, “the truth should be taught by one sentence”. This sentence is exactly the one-line determinism. How happy would I be if I could read this book when I was in junior or high school! I still feel that I am so lucky to have the opportunity to see that big picture of calculus.

2、An old teacher’s comments：

This is the finishing touch, which directly grabs the key to the fundamental formula of the calculus, i.e. (1) determined by “finding the sub-height” and (2) in place by “finding the total height". Along with Qun Lin’s “height-calculation figure for calculus” (i.e. Fig. 2-24). The whole big idea of calculus will appear clearly to people immediately. Now I understand the meaning of “the way to achieve great deed is usually the simplest way!” The "mean value theorem" that people often talk about is actually influenced by the rigid and ideological adherence to discredited ideas. You can imagine the serious mental torture suffered by people like me with 35-years experience of teaching calculus. You said: “to learn mathematics to practice the ability of getting the big idea." This is very vivid and correct!