THE FORMATION AND DECAY OF SHOCK WAVES

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1. Introduction. The theory of propagation of shock waves is one of a small class of mathematical topics whose basic problems are easy to explain but hard to resolve. This article is a brief introduction to the subject: we shall describe the origin of the governing equations, some of the striking phenomena, and a few of the mathematical tools used to analyse them.

2. What is a conservation law? A conservation law asserts that the change in the total amount of a physical entity contained in any region G of space is due to the **flux** of that entity across the boundary of G. In particular, the rate of change is

$$\frac{d}{dt}\int_G u\,dx = -\int_{\partial G} f\cdot n\,dS, \tag{2.1}$$

where u measures the **density** of the physical entity under discussion, and the vector f describes its flux; n is the outward normal to the boundary ∂G of G. If u and f are differentiable functions, we can, on the left, perform the differentiation under the integral sign and on the right apply the divergence theorem. We obtain

$$\int_G \{u_t + \operatorname{div} f\}\,dx = 0.$$

This relation is assumed to be valid for every domain G. Letting G shrink to a point and dividing by the volume of G we get the differential form of the conservation law:

$$u_t + \operatorname{div} f = 0. \tag{2.2}$$

To complete the theory we need some law relating f to u. E.g., Newton's law of cooling asserts that the flux of heat is proportional to the negative gradient of u, where u is temperature; in this case $f = -h$ grad u, h positive, so (2.2) becomes

$$u_t - h\Delta u = 0, \qquad \Delta = \text{div grad}.$$

In this example f depends on the derivatives of u; in what follows *we assume that f depends on u alone.* More precisely, we shall be looking at systems of conservation laws

$$u_t^j + \operatorname{div} f^j = 0, \qquad j = 1, \cdots, n, \tag{2.3}$$

where each f^j is a function of all the $u^1, \cdots, u^n$, and a nonlinear function at that.

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Many equations of mathematical physics are of this form, in particular, those governing the flow of a nonviscous, compressible fluid.

We shall concern ourselves with the **initial value problem** for systems of form (2.3); that is, given the value of each u^j at $t = 0$ as function of x, determine u^j as function of x and t for all $t > 0$.

3. The theory of a single nonlinear conservation law. In this section we shall study conservation laws for a single quantity u dependent on only one space variable x; in this case f has only one component:

$$u_t + f_x = 0, \tag{3.1}$$

where f is some nonlinear function of u. Denoting

$$\frac{df}{du} = a(u) \tag{3.2}$$

we can write (3.1) in the form

$$u_t + a(u)u_x = 0 \tag{3.3}$$

which asserts that u is constant along trajectories $x = x(t)$ which propagate with speed a:

$$\frac{dx}{dt} = a. \tag{3.4}$$

For this reason a is called the **signal speed**; the trajectories, satisfying (3.4), are called **characteristics.** Note that if f is a nonlinear function of u, both signal speed and characteristics depend on the solution u.

The constancy of u along characteristics combined with (3.4) shows that the characteristics propagate with constant speed; so they are straight lines. This leads to the following geometric solution of the initial value problem

$$u(x, 0) = u_0(x).$$

Draw straight lines issuing from points y of the x-axis, with slope $1/u_0(y)$ (see Fig. 1).

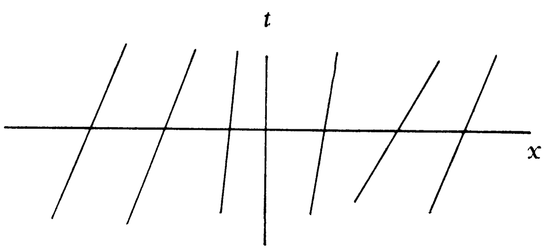

FIG. 1

As we shall show, if u_0 is a C^1 function, these lines simply cover a neighborhood of the x-axis; since the value of u along the line issuing from the point y is $u_0(y)$, $u(x,t)$ is uniquely determined near the x-axis.

An analytical form of this construction goes like this (see Fig. 2)

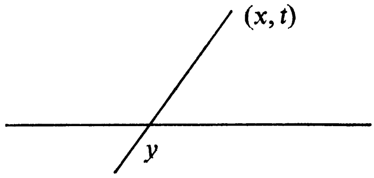

FIG. 2

Let (x,t) be any point, y the intersection of the characteristic through x, t with the x-axis. Then $u = u(x,t)$ satisfies

$$u = u_0(y),\ y = x - t\,a(u). \tag{3.5}$$

Assume u_0 differentiable; then, according to the implicit function theorem, (3.5) can be solved for u as a differentiable function of x and t for t small enough, and

$$u_t = -\frac{u_0'a}{1 + u_0'a_u t} \qquad u_x = \frac{u_0'}{1 + u_0'a_u t}. \tag{3.6}$$

Substituting (3.6) into (3.3) we see immediately that u defined by (3.5) satisfies (3.3).

Let's assume that equation (3.3) is **genuinely nonlinear**, i.e., that $a_u \neq 0$ for all u, say

$$a_u > 0. \tag{3.7}$$

Then if u_0' is $\geqq 0$ for all x, u_t and u_x as given by formulas (3.6) remain bounded for all $t > 0$; on the other hand, if u_0' is < 0 at some point, both u_t and u_x tend to ∞ as $1 + u_0'\ a_u(u_0)t$ approaches zero. Both these facts can be deduced from the geometric form of the solution contained in Figure 1:

In the first case, when $u_0(x)$ is an increasing function of x, the characteristics issuing from the x-axis diverge in the positive t direction, so that the characteristics simply cover the whole half-plane $t > 0$. In the second case there are two points y_1 and y_2 such that $y_1 < y_2$, and $u_1 = u_0(y_1) > u_0(y_2) = u_2$; then by (3.7) also $a_1 = a(u_1) > a(u_2) = a_2$ so that the characteristics issuing from these points intersect at time

$$t = \frac{y_2 - y_1}{a_1 - a_2}.$$

At the point of intersection, u has to take on the value u_1 and u_2 both, an impossibility (see Fig. 3).

Both the geometric and the analytic argument prove beyond the shadow of a

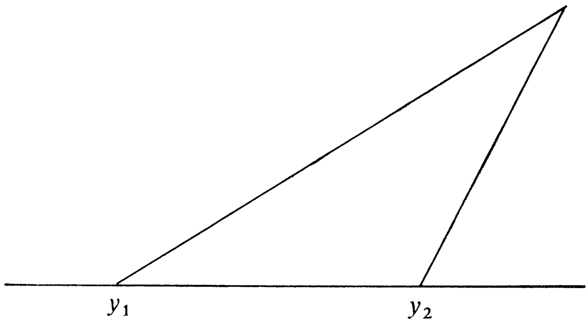

FIG. 3

doubt that if the initial value u_0 is not an increasing function of x then *no continuous function* $u(x,t)$ *exists for all* $t > 0$ *with initial value* u_0 *which solves equation* (3.3) *in the ordinary sense*!

What happens after continuous solutions cease to exist? After all, the world does not come to an end. For an answer, we turn to experiments with compressible fluids: these clearly show the appearance of discontinuities in solutions. We begin our study of discontinuous solutions with the simplest kind, those satisfying (3.1) in the ordinary sense on each side of a smooth curve $x = y(t)$ across which u is discontinuous. We shall denote by u_l and u_r the values of u on the left and right sides respectively of $x = y(t)$. Choose a and b so that the curve y intersects the interval $a \leqq x \leqq b$ at time t (see Fig. 4).

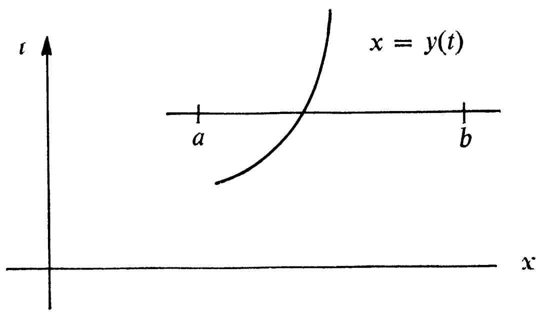

FIG. 4

Denoting by $I(t)$ the quantity $I(t) = \int_a^b u(x,t)\,dx = \int_a^y + \int_y^b$, we have

$$\frac{dI}{dt} = \int_a^y u_t dx + u_l s + \int_y^b u\,dx - u_r s, \tag{3.8}$$

where we have used the abbreviation

$$s = \frac{dy}{dt} \tag{3.9}$$

for the speed with which the discontinuity propagates. Since on either side of the discontinuity (3.1) is satisfied we may set $u_t = -f_x$ in the integrals in (3.8); after carrying out the integration we obtain $dI/dt = f_a - f_l + u_l s - f_b + f_r - u_r s$; here we have used the handy abbreviations

$$f(u_l) = f_l, \qquad f(u_r) = f_r$$
$$f(u(a)) = f_a, \qquad f(u(b)) = f_b.$$

The conservation law asserts that $dI/dt = f_a - f_b$. Combining this with the above relation we deduce the **jump condition**

$$s[u] = [f], \tag{3.10}$$

where $[u] = u_r - u_l$ and $[f] = f_r - f_l$ denote the jump in u and in f across y.

We show now in an example that previously unsolvable initial value problems can be solved for all t with the aid of discontinuous solutions. Take

$$f(u) = \tfrac{1}{2}u^2, \tag{3.11}$$

$$u_0(x) = \begin{cases} 1 & \text{for } x \leqq 0 \\ 1 - x & \text{for } 0 \leqq x \leqq 1 \\ 0 & \text{for } 1 \leqq x. \end{cases}$$

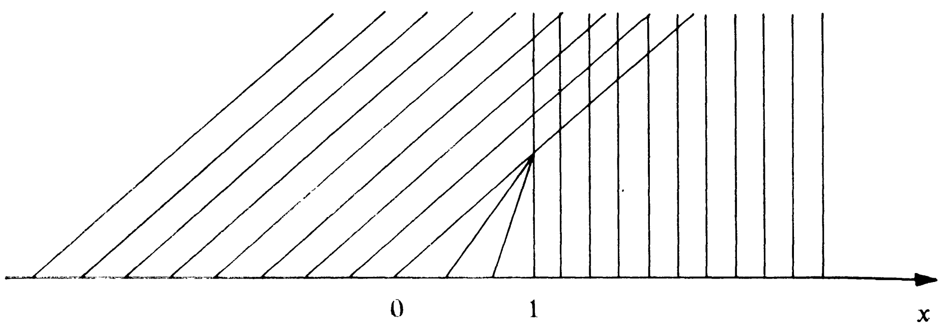

FIG. 5

The geometric solution is single valued for $t \leqq 1$ but double valued thereafter (see Fig. 5). Now we define for $t \geqq 1$

$$u(x,t) = \begin{cases} 1 \text{ for } x < (1+t)/2 \\ 0 \text{ for } (1+t)/2 < x. \end{cases}$$

The discontinuity starts at (1,1); it separates the state $u_l = 1$ on the left from the state $u_r = 0$ on the right; the speed of propagation was chosen according to the jump condition (3.10), with $f(u) = \frac{1}{2}u^2$:

$$s = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}.$$

Introducing generalized solutions makes it possible to solve intial value problems which could not be solved within the class of genuine solutions. At the same time there is the danger that the enlarged class of solutions is so large that there are several generalized solutions with the same initial data. The following example shows that this anxiety is well founded:

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 < x. \end{cases}$$

The geometric solution

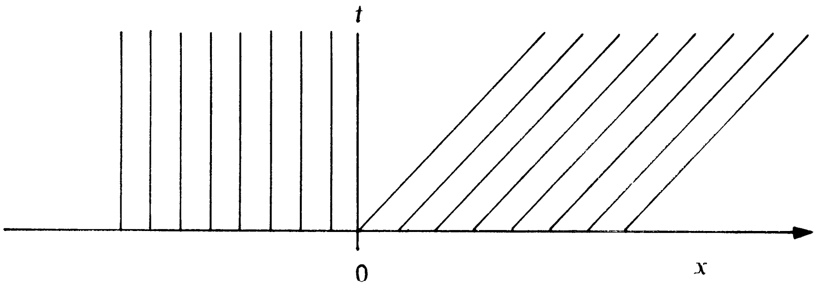

FIG. 6

is single valued for $t > 0$ (see Fig. 6) but does not determine the value of u in the wedge $0 < x < t$. We could fill this gap in the fashion of the previous example and set

$$u(x,t) = \begin{cases} 0 & \text{for } x < t/2 \\ 1 & \text{for } t/2 < x. \end{cases} \tag{3.12}$$

The speed of propagation was so chosen that the jump condition (3.10) is satisfied. On the other hand the function

$$u(x,t) = x/t, \qquad 0 \leqq x \leqq t \tag{3.12$'$}$$

satisfies the differential equation (3.3) with $a(u) = u$, and joins continuously the rest of the solution determined geometrically. Clearly only one of these solutions can have physical meaning; the question is which?

We reject the discontinuous solution (3.12) for failure to satisfy the following criterion:

The characteristics starting on either side of the discontinuity curve when continued in the direction of positive t intersect the line of discontinuity. This will be the case if

$$a(u_l) > s > a(u_r). \tag{3.13}$$

Under condition (3.7) for a this means that

$$u_l > u_r. \tag{3.14}$$

Clearly this condition is violated in the solution given by (3.12).

The analysis at the beginning of this section shows that signals propagate along characteristics. Condition (3.13) allows each point of the discontinuity to be reached by characteristics on both sides, so that the shock is influenced by the initial data of the solution; this constitutes one justification of Condition (3.13). Another justification can be based on characterising the physically meaningful solutions as limits, when u tends to zero, of the viscous equation

$$u_t + f(u)_x = \mu u_{xx}, \qquad \mu > 0.$$

Yet another justification can be based on the theory of entropy. We shall not go into this interesting matter any deeper here, but merely record the gratifying fact that when $a(u)$ is a monotonic function of u, condition (3.13) is restrictive enough to make the solution of the initial value problem unique, yet it is broad enough to allow the construction of a solution for all time $t > 0$, having as initial value any integrable function u_0. True, the concept of solution has to be generalized beyond simple discontinuities: a bounded measurable function $u(x,t)$ is said to satisfy the conservation law (3.1) in the sense of distributions, if for all continuously differentiable test functions $\phi(x,t)$, with support in $t > 0$,

$$\int\int [\phi_t u + \phi_x f(u)]dxdt = 0. \tag{3.15}$$

It is easy to verify that for the previously considered class of piecewise continuous solutions condition (3.15) is equivalent with the jump condition (3.10).

For merely bounded, measurable solutions u_l and u_r in condition (3.13) have to be interpreted as follows:

$$\begin{aligned} u_l &= \liminf_{y \to x,\ y<x} u(y,t), \\ u_r &= \limsup_{y \to x,\ x<y} u(y,t). \end{aligned}$$

For the main existence theorem we refer the reader to [**8**] and [**13**], and for uniqueness to [**1**], [**14**], and [**16**].

It turns out that when $a(u)$ is not monotonic, condition (3.13) is not sufficient to guarantee unique determination of solutions by their initial data. A replacement for this condition has been found by Oleinik; this condition, together with the existence and uniqueness theorem is described in [**15**]; other interesting discussions of this condition are contained in [**4**], [**6**], and [**16**].

4. The decay of solutions. Existence and uniqueness of solutions is not the end but merely the beginning of a theory of differential equations. The really interesting questions concern the behavior of solutions.

Here we shall study the asymptotic behavior for large time of solutions of conservation laws of form (3.1) which satisfy condition (3.14); we assume that $a(u)$ is an *increasing* function of u.

As remarked in Section 3, any differentiable solution u is constant along characteristics

$$\frac{dx}{dt} = a(u) = f'(u). \tag{4.1}$$

Let $x_1(t)$ and $x_2(t)$ be a pair of characteristics, $0 \leqq t \leqq T$. Then there is a whole one-parameter family of characteristics connecting the points of the interval $[x_1(0), x_2(0)]$, $t = 0$ with points of the interval $[x_1(T), x_2(T)]$, $t = T$; since u is constant along these characteristics, *$u(x,0)$ on the first interval and $u(x,T)$ on the second interval are equivariant*, i.e., they take on the same values in the same order. Since equivariant functions have the same total increasing and decreasing variations, we conclude that *the total increasing and decreasing variations of a differentiable solution between any pair of characteristics are conserved.*

Denote by $D(t)$ the width of the strip bounded by x_1 and x_2:

$$D(t) = x_2(t) - x_1(t) > 0. \tag{4.2}$$

Differentiating (4.2) with respect to t and using (4.1), we get

$$\frac{d}{dt}D(t) = \frac{dx_2}{dt} - \frac{dx_1}{dt} = a(u_2) - a(u_1). \tag{4.3}$$

Integrating with respect to t we get

$$D(T) = D(0) + [a(u_2) - a(u_1)]T. \tag{4.4}$$

Suppose there is a shock y present in u between the characteristics x_1 and x_2 (see Fig. 7). Since according to condition (3.13) characteristics on either side of a shock run into the shock, there exist for any given time T two characteristics y_1 and y_2 which intersect the shock y at exactly time T. Assuming that there are no other shocks present we conclude that the increasing variation of u on $(x_1(t), y_1(t))$, as well as on $(x_2(t), y_2(t))$, is independent of t. According to condition (3.14), u decreases across shocks, so the increasing variation of u along $[z_1(T), x_2(T)]$ equals the sum of the increasing variations of u along $[x_1(0), y_1(0)]$ and along $[y_2(0), x_2(0)]$. This sum is in general less than the increasing variation of u along $[x_1(0), x_2(0)]$, therefore we conclude that if shocks are present, *the total increasing variation of u between two characteristics decreased with time.*

We give now a quantitative estimate of this decrease. Let I_0 be any interval of the x-axis; we subdivide it into subintervals $[y_{j-1}, y_j]$, $j = 1, \cdots, n$ in such a way

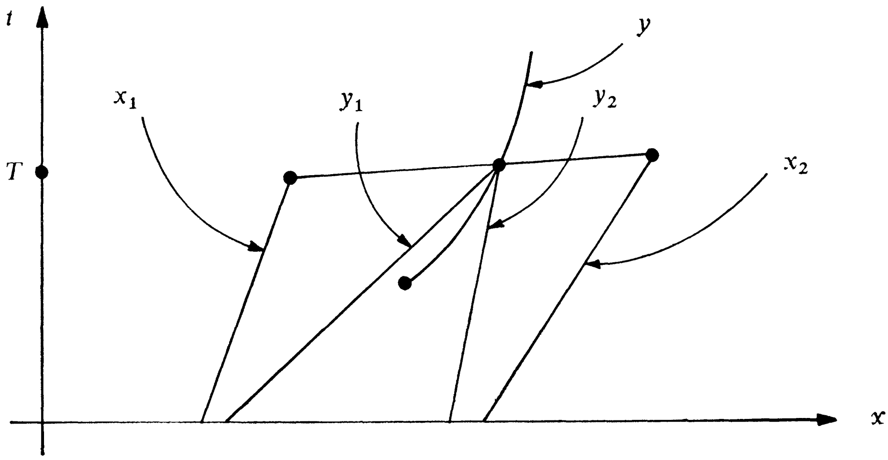

FIG. 7

that $u(x, 0)$ is alternately increasing and decreasing on the intervals (we here assumed for simplicity that u_0 is piecewise monotonic). We denote by $y_j(t)$ the characteristic issuing from the jth point y_j, with the understanding that if $y_j(t)$ runs into a shock, $y_j(t)$ is continued as that shock.

It is easy to show that for any $t > 0$, $u(x, t)$ is alternately increasing and decreasing on the intervals $(y_{j-1}(t),\ y_j(t))$. Since a is an increasing function of u, and since according to (3.14) u decreases across shocks, the total increasing variation $A^+(T)$ of $a(u)$ across the interval $I(T) = [y_0(T),\ y_n(T)]$ is

$$\sum_{j \text{ odd}} a(u_j(T) - a(u_{j-1}(T)) = A^+(T), \tag{4.5}$$

where $u_{j-1}(T)$ denotes the value of u on the right edge of $y_{j-1}(T)$, $u_j(T)$ denotes the value of u on the left edge of $y_j(T)$; in case $y_{j-1}(T)$ and $y_j(T)$ are the same, the jth term in (4.5) is zero. Suppose $y_{j-1}(T)$ and $y_j(T)$ are shocks; then there exist characteristics $x_{j-1}(t)$ and $x_j(t)$ which start at $t = 0$ inside (y_{j-1}, y_j) and which at $t = T$ run into $y_{j-1}(T)$ and $y_j(T)$ respectively. The value of u along $x_j(t)$ is $u_j(T)$.

Denote $x_j(t) - x_{j-1}(t)$ by $D_j(t)$; according to (4.4)

$$D_j(T) = D_j(0) + [a(u_j) - a(u_{j-1})]T.$$

Summing over j odd and using (4.5) we get

$$\Sigma D_j(T) = \Sigma D_j(0) + A^+(T)T. \tag{4.6}$$

Since the intervals $[x_{j-1}(T), x_j(T)] = [y_{j-1}(T), y_j(T)]$ are disjoint and lie in $I(T)$, their total length cannot exceed the length $L(t)$ of $I(T)$; so we deduce from (4.6) that

$$A^+(T) \leqq \frac{L(T)}{T}, \tag{4.7}$$

where $A^+(T)$ is the total increasing variation of $a(u)$ along $I(T)$.

Let $u(x,t)$ be a solution of (3.1), possibly discontinuous, whose initial values are bounded, and zero outside a finite interval I_0. Since signals propagate with finite speed, for every t the solution $u(x,t)$ is zero outside some finite x-interval $I(t)$. Denote by $v(t)$ and $w(t)$ the values of u at the left and right endpoints of $I(t)$ respectively. Since the endpoints may lie on shocks, these values need not be zero, however it follows from (3.14) that

$$v(t) \leqq 0, \qquad 0 \leqq w(t). \tag{4.8}$$

Denote by s_{left} and s_{right} the speed with which the shocks at the endpoints propagate; according to the jump relation (3.10).

$$s_{\text{left}} = \frac{f(v)-f(0)}{v}, \quad s_{\text{right}} = \frac{f(w)-f(0)}{w}. \tag{4.9}$$

Since a is an increasing function of u, $f(u)$ is convex. It follows from the mean value theorem that the difference quotient of f over an interval is not less than f' at the left endpoint, and not greater than f' at the right endpoint of that interval. So it follows from (4.8) that

$$a(v) \leqq \frac{f(v)-f(0)}{v}, \quad \frac{f(w)-f(0)}{w} \leqq a(w). \tag{4.10}$$

At this point we assume that a is strictly increasing, i.e., that for some positive number k

$$0 < k \leqq a'; \tag{4.11}$$

here we abbreviate d/du by prime. It follows that inequalities (4.10) are strict; combining these with (4.9) we can put them into this form

$$s_{\text{right}} - s_{\text{left}} \leqq \theta[a(w) - a(v)], \tag{4.12}$$

where θ *is* < 1.

Denote the length of $I(t)$ by $L(t)$; since s_{left} and s_{right} are the speeds with which the endpoints of I move,

$$\frac{dL}{dt} = s_{\text{right}} - s_{\text{left}}. \tag{4.13}$$

Substituting the inequalities (4.12) into (4.13) we get

$$\frac{dL}{dt} \leqq \theta[a(w) - a(v)].$$

Since by (4.8) $v < w$, $a(w) - a(v)$ is bounded by the total increasing variation $A^+(t)$ of $a(u)$ over $I(t)$:

$$a(w) - a(v) \leqq A^+(t). \tag{4.14}$$

Combining the last two inequalities we get

$$\frac{dL}{dt} \leqq \theta A^+(t).$$

Using inequality (4.7) we get

$$\frac{dL}{dt} \leqq \frac{\theta}{t} L(t);$$

and multiplying by $t^{-\theta}$ we deduce that

$$\frac{d}{dt}(t^{-\theta}L) \leqq 0.$$

Thus $t^{-\theta}L(t)$ is a decreasing function of time; in particular

$$L(t) \leqq t^{\theta}L(1) \qquad \text{for } t > 1. \tag{4.15}$$

Substituting this into the right side of (4.7) we get

$$A^+(t) \leqq t^{\theta-1}L(1).$$

Since $\theta < 1$, this shows that $A^+(t) \to 0$ as $t \to \infty$.

It follows from the strictly increasing character (4.11) of $a(u)$ that the total increasing variation of u along $I(t)$ is bounded by $A^+(t)/k$. Since u is $\leqq 0$ at the left endpoint of $I(t)$ and $\geqq 0$ at the right endpoint, it follows that likewise *the maximum* $m(t)$ *of* $u(x,t)$ *over* $I(t)$ *is bounded by* $A^+(t)/k$;

$$m(t) \leqq A^+(t)/k. \tag{4.16}$$

Combining this with the above estimate for A^+ we get that $m(t) \leqq$ const $t^{\theta-1}$ which shows that *the maximum of u at time t tends to zero like* $t^{\theta-1}$.

This result is somewhat crude; a more detailed analysis will furnish a more precise result. (A different derivation was given by Barbara Quinn in her dissertation at New York University, 1970.) We start by expressing $f(r)$, $f(w)$ in (4.9) by their Taylor expansions; we get

$$\begin{aligned} s_{\text{left}} &= f'(0) + \frac{1}{2}f''(0)v + \frac{1}{6}f'''(\bar{v})v^2, \\ s_{\text{right}} &= f'(0) + \frac{1}{2}f''(0)w + \frac{1}{6}f'''(\bar{w})w^2, \end{aligned} \tag{4.17}$$

where $v < \bar{v} < 0$, $0 < \bar{w} < w$.

Denote by K an upper bound for f'''; since m is an upper bound for $|v|$ and w, it follows that

$$s_{\text{left}} \geqq f'(0) + \frac{1}{2}\left[f''(0) + \frac{K}{3}m\right]v$$

$$s_{\text{right}} \leqq f'(0) + \frac{1}{2}\left[f''(0) + \frac{K}{3}m\right]w.$$

Substituting this into (4.13) we get

$$\frac{dL}{dt} \leqq \frac{1}{2}\left[f''(0) + \frac{K}{3}m\right](w - v). \tag{4.18}$$

It follows from (4.11) and (4.14) that

$$w - v \leqq \frac{a(w) - a(v)}{k} \leqq \frac{A^+(t)}{k}. \tag{4.19}$$

The constant k in (4.11) has to be a lower bound of $a' = f''(u)$ for $|u| \leqq m$; in particular we can take

$$k = f''(0) - Km. \tag{4.20}$$

Substituting this into (4.19) and then into (4.18) we get that for m small enough

$$\frac{dL}{dt} \leqq \frac{1}{2}\left[\frac{f''(0) + K/3\,m}{f''(0) - Km}\right]A^+ \leqq \left(\frac{1}{2} + Hm\right)A^+. \tag{4.21}$$

We substitute into (4.21) estimate (4.16) for m, and then estimate (4.7) for A^+; we obtain the following inequality:

$$\frac{dL}{dt} \leqq \left(\frac{1}{2} + \frac{H}{k}\frac{L}{t}\right)\frac{L}{t}. \tag{4.22}$$

Introduce a new variable J by $L = J\sqrt{t}$; (4.22) becomes

$$\sqrt{t}\,\frac{dJ}{dt} \leqq \frac{H}{k}\frac{J^2}{t}.$$

Dividing by $\sqrt{t}\,J^2$ we get, after integrating from T to $t > T$, that

$$\frac{1}{J(T)} - \frac{1}{J(t)} \leqq \frac{H}{2k}\left(\frac{1}{\sqrt{T}} - \frac{1}{\sqrt{t}}\right),$$

which implies that

$$\frac{1}{J(T)} - \frac{H}{2k\sqrt{T}} \leqq \frac{1}{J(t)}. \tag{4.23}$$

According to (4.15), $L(T)/T = J(T)/\sqrt{T}$ tends to 0 as $T \to \infty$; this implies that for T large enough, the left side of (4.23) is positive. Then (4.23) furnishes an upper bound for $J(t)$ for all $t > T$. The boundedness of $J(t)$ implies that $L(t)$ is $O(\sqrt{t})$ as $t \to \infty$. Combining this with the estimates (4.7) and (4.16) we reach the following conclusion.

THEOREM 4.1. *Let u be a possibly discontinuous solution of the conservation law $u_t + f_x = 0$, where f is three times differentiable and strictly convex. Suppose that all discontinuities of u satisfy* (3.13), *and that $u(x,0)$ has compact support. Then*

(a) *the length of the support of $u(x,t)$ is $O(\sqrt{t})$,*

(b) $\operatorname{Max}_x |u(x,t)| = O(1/\sqrt{t})$.

It turns out that this result is rather precise: Using an explicit formula one can show, see [9], that the length of the support of u divided by $\sqrt{t}$ tends to a limit, and so does $\sqrt{t}\ \operatorname{Max} |u|$.

We turn now to solutions which are **periodic** in x:

$$u(x + p, t) = u(x,t).$$

We take $I(T)$ to be any interval of length p at time T. According to our basic estimate (4.7), the increasing variation of $a(u)$ per period is $\leqq p/T$. It follows then from (4.11) that the increasing variation per period of u itself does not exceed p/kT. Since u is periodic, its decreasing and increasing variations are equal, and serves as bound for the oscillation of u, in particular for the deviation of u from its mean value per period.

For a periodic solution $u(x,t)$, the flux f at $(0,t)$ equals the flux at (p,t); thus the total flux into an interval of length p is zero, and so the mean value of u,

$$\bar{u} = \frac{1}{p} \int_0^p u(x,t)dx,$$

is independent of t. We summarize our results as follows:

THEOREM 4.2. *Let $u(x,t)$ be a possibly discontinuous solution of $u_t + f_x = 0$, f strictly convex, $f'' > k > 0$. Suppose that all discontinuities of u satisfy* (3.13) *and that u is periodic in x with period p. Then*

(a) *The total variation of u at time t does not exceed $2p/kt$,*

(b)

$$|u(x,t) - \bar{u}| \leqq 1/kt, \tag{4.24}$$

where $\bar{u}$ is the mean value of u.

Again it can be shown that (4.24) is sharp, i.e., that

$$\lim_{t \to \infty} t \max_x |u(x,t) - \bar{u}| = k = f''(\bar{u}). \tag{4.25}$$

The surprising, almost paradoxical feature of inequality (4.24) is that it holds uniformly for all solutions with period p; it is independent of the amplitude of the initial disturbance. All that the initial amplitude can influence is the time when the asymptotic estimate (4.24) becomes accurate: The *larger* the initial amplitude, the *sooner* (4.25) converges. This is in sharp contrast to the linear case where the asymptotic amplitude of a signal for large time is proportional to its initial amplitude, but the time it takes to reach the asymptotic shape is independent of the initial amplitude.

Let $u_1(x)$ be an initial function which is zero outside the interval $[0, p]$, and define $u_2(x)$ to be equal $u_1(x)$ in $[0, p]$, and periodic (see Fig. 8).

According to Theorem 4.1, $u_1(x, t)$ decays like $1/\sqrt{t}$; $u_2(x, t)$ on the other hand is periodic,[1] so its asymptotic behavior is governed by Theorem 4.2: $u_2(x, t)$ decays like $1/t$. So we have the paradoxical result that u_2, which represents a much larger initial disturbance than u_1, nevertheless decays faster than u_1.

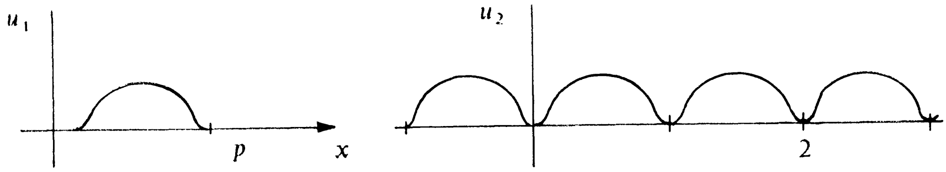

FIG. 8

5. Systems of conservation laws. Models which are at all realistic are governed by a whole system of conservation laws, rather than by a single one. The value of what we have learned about single equations lies in the light this knowledge sheds on systems. It turns out that the main phenomena we have found: the breakdown of continuous solutions, the necessity of imposing an entropy-like condition to distinguish those discontinuous solutions which are physically realizable from those which are not, and the decay of solutions as $t \to \infty$, have their counterparts for systems. That is not to say that the theory is as far advanced for systems as it is for single equations; on the contrary, what we have is a sea of conjectures, confined partly by the shores of numerical computations, with a few islands of solidly proved mathematical facts.

What are the proven facts about systems? In [**10**] the author has shown that solutions of 2×2 systems of conservation laws break down after a finite time, unless the initial data satisfy a monotonicity condition. In [**9**], an analogue of the entropy condition (3.13) is described, and a condition for genuine nonlinearity is given. In [**15**], Oleinik gives a uniqueness theorem for solutions of systems of two conservation laws of which one is linear. In [**2**], Glimm solves the initial value problem for systems, for initial data with small oscillation. In [**5**], Johnson and Smoller solve the initial value problem for initial data which satisfy a certain monotonicity condition, for 2×2 systems which satisfy a certain convexity-like condition. The only existence

[1] Solutions whose initial values are periodic are periodic for all t; this follows from the uniqueness theorem that solutions which are equal at $t = 0$ are equal for all $t > 0$.

theorem with no restrictions on the initial data is due to Nishida, [12], and works only for the system

$$u_t + v_x = 0, \qquad v_t - \left(\frac{1}{u}\right)_x = 0.$$

In [3], Glimm and the author prove the decay of solutions with small oscillation of 2×2 systems. The method described in Section 4 is taken from that paper.

For those who wish to work in this field I recommend Glimm's paper [2]. It contains a wealth of ideas, such as the use of an approximation scheme containing a sequence of random parameters; the scheme is shown to converge for almost all values of the parameters. Glimm also introduces novel, nonlocally defined functionals; the estimate of the growth and decay of these functionals plays a crucial role in the existence theorem.

This article is an expanded version of an invited address delivered at the January 1970 meeting of the MAA at San Antonio, Texas. Other versions of this talk were given at Oregon State University, Corvallis; Texas Tech. University, Lubbock, and at Brown University. The talk is partly based on the joint paper [3] with James Glimm.

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