Uncertainty Quantification for Geometric Registration

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Introduction

- we propose a local uncertainty quantification framework for multi-scan registration, Geometrically Stable Sampling, Uncertainty Visualization and Model-based View Planning.
- We consider two formulations: Joint Pairwise Registration (JPR) and Simultaneous Registration and Reconstruction (SRAR).

Formulations

Formulation of Joint Pairwise Registration (JPR).

$$\underset{\{T_i\}}{\text{minimize}} \sum_{(i,i')\in\mathcal{E}} d^2(S_i, T_i, S_{i'}, T_{i'}) \quad \text{subject to} \quad R_1 = I_3, t_1 = \mathbf{0}. \tag{1}$$

Formulation of Simultaneous Registration and Reconstruction (SRAR).

$$\underset{\{R_i,t_i\},\{(d_k,n_k)\}}{\text{argmin}} \sum_{i=1}^{N}\sum_{j=1}^{N_i} \left((R_i p_{ij} + t_i)^T n_{k_{ij}} - d_{k_{ij}}\right)^2 \quad \text{subject to} \quad R_1 = I_3, t_1 = \mathbf{0}. \tag{2}$$

Uncertainty Quantification

Notations.

- Underlying ground truth scan points: p^{gt}.
- Input Noise: x
- Output Uncertainty: y.
- Hyper-parameters: w.
- Registration Error: $f(x, y, w, p^{\text{gt}})$.

Multi-scan Registration as an Optimization problem.

$$x^\star(y, w, p^{\text{gt}}) = \underset{x}{\text{argmin}}\, f(x, y, w, p^{\text{gt}}). \tag{3}$$

Linear Map Approximation.

$$x^\star(y, w, p^{\text{gt}}) \approx x^\star(0) + \frac{\partial x^\star}{\partial y}(y, w, p^{\text{gt}}) \cdot y = \frac{\partial x^\star}{\partial y}(0, w, p^{\text{gt}}) \cdot y. \tag{4}$$

Replacing the underlying ground-truth.

$$x^\star(y, w, p^{\text{gt}}) \approx \frac{\partial x^\star}{\partial y}(0, w, p^\star) \cdot y. \tag{5}$$

Approximate covariance matrix estimation. If $E[y] = 0$, $\frac{\partial x^\star}{\partial y}(0, w, p^\star)$ can be approximated by a Gaussian distribution with zero mean. Its covariance matrix is denoted by

$$C(w, p^\star) \approx \mathbb{V}[x^\star(y, w, p^{\text{gt}})]$$

This covariance matrix can be approxiamted by

$$C(w, p^\star) := \frac{\partial x^\star}{\partial y}(0, w, p^\star) \cdot \mathbb{V}[y] \cdot \frac{\partial x^\star}{\partial y}(0, w, p^\star)^T. \tag{6}$$

Model-based View Planning

Weighted SRAR. Assumes n viewpoints, each with weight $w_i \in \{0, 1\}$.

$$f_{\text{weighted}} := \sum_{i=1}^{n} w_i \sum_{j=1}^{n_i} \left((R_i p_{ij} + t_i)^T n_{k_{ij}} - d_{k_{ij}}\right)^2 \tag{7}$$

Uncertainty Score. Uncertainty can be quantified as trace of sub-covariance matrix.

$$C(w) := \left(\sum_{i=1}^{n} w_i G_i G_i^T\right)^{-1} \left(\sum_{i=1}^{n} w_i^2 G_i G_i^T\right) \left(\sum_{i=1}^{n} w_i G_i G_i^T\right) \tag{8}$$

$$s_{\text{uncertainty}}(w) := \text{Trace}([C(w)]_{\text{Pose}}). \tag{9}$$

Objective Function. Introduce balance between length and uncertainty.

$$\min_{w\in\{0,1\}^n} f_{\text{uncertainty}}(w) + \lambda f_{\text{length}}(w) \tag{10}$$

Analysis of Approximation Error

Proposition 1 Given w, p, the derivative $\frac{\partial x^\star}{\partial y}(\mathbf{0}, w, p)$ admits the following expression:

$$\frac{\partial x^\star}{\partial y}(\mathbf{0}, w, p) := -\left(\frac{\partial^2 f}{\partial x^2}(\mathbf{0}, \mathbf{0}, w, p)\right)^{-1} \left(\frac{\partial^2 f}{\partial x \partial y}(\mathbf{0}, \mathbf{0}, w, p)\right) \tag{11}$$

Theorem 1. In this theorem we provide an error bound for approximate expression . Suppose $x^\star(y)$ is implicit function determined by $x^\star(y) = \arg\min_x f(x, y, \bar{r})$ where $y \in \mathbb{R}^d$. Let

$$V = \frac{\partial x^\star}{\partial y}(\mathbf{0}, w, \bar{r}) \cdot \text{Var}[y] \cdot \frac{\partial x^\star}{\partial y}(\mathbf{0}, w, \bar{r})^T$$

then the Taylor's theorem gives that

$$|\text{Var}[x^\star(y)] - V| \le \sum_{k=2}^{\infty} \frac{1}{k!} \|(x^2)^{(k)}(\mathbf{0})\|_S \delta^k \sigma^k$$

where the norm $\|T\|_S$ is defined as

$$\|f^{(k)}\|_S := \sum_{\alpha} \binom{d}{\alpha} f^{(k)}(\alpha).$$

in which $\alpha \in \mathbb{N}^d$ and we employ the definition

$$\binom{d}{\alpha} = \frac{d!}{\prod \alpha_i!}$$

$$f^{(k)}(\alpha) = \frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

where $x_1, \dots, x_n$ are n variables of function f, $\alpha \in \mathbb{N}^n$ and $\sum_i \alpha_i = k$.

Proposition 2 In this proposition we show that whenever the derivatives of f decay in geometric rates then $\|x^{(k)}(\mathbf{0})\|_S$ would not grow faster than exponential. Suppose

$$|f_{xx}| \ge B, |f_{x^l y^s}| \le C r^{l+s}$$

holds for some constant C and r, then $|x^{(k)}| \le C' k! v^k$ for some constant C' and v.

Theorem 2. The objective functions of JPR and SRAR both satisfy the conditions in Proposition 2.

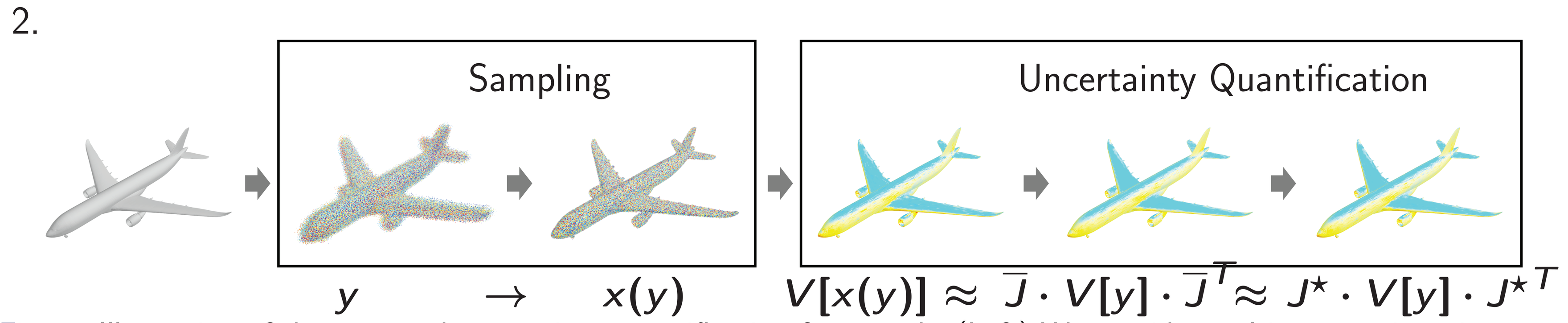

Figure: Illustration of the proposed uncertainty quantification framework. (Left) We consider multi-scan registration as a mapping between a set of input scans and optimized scan poses using an off-the-shelf algorithm. The input scans are generated from a fixed set of unknown camera poses through a noise model characterized by parameter y. (Right) Our uncertainty quantification approach approximates the covariance of the output in two steps. The first step leverages an approximated linear map of the non-linear map at the underlying ground-truth. The second step uses the current optimal solution to approximate the Jacobi matrix in the linear map.

Uncertainty Visualization

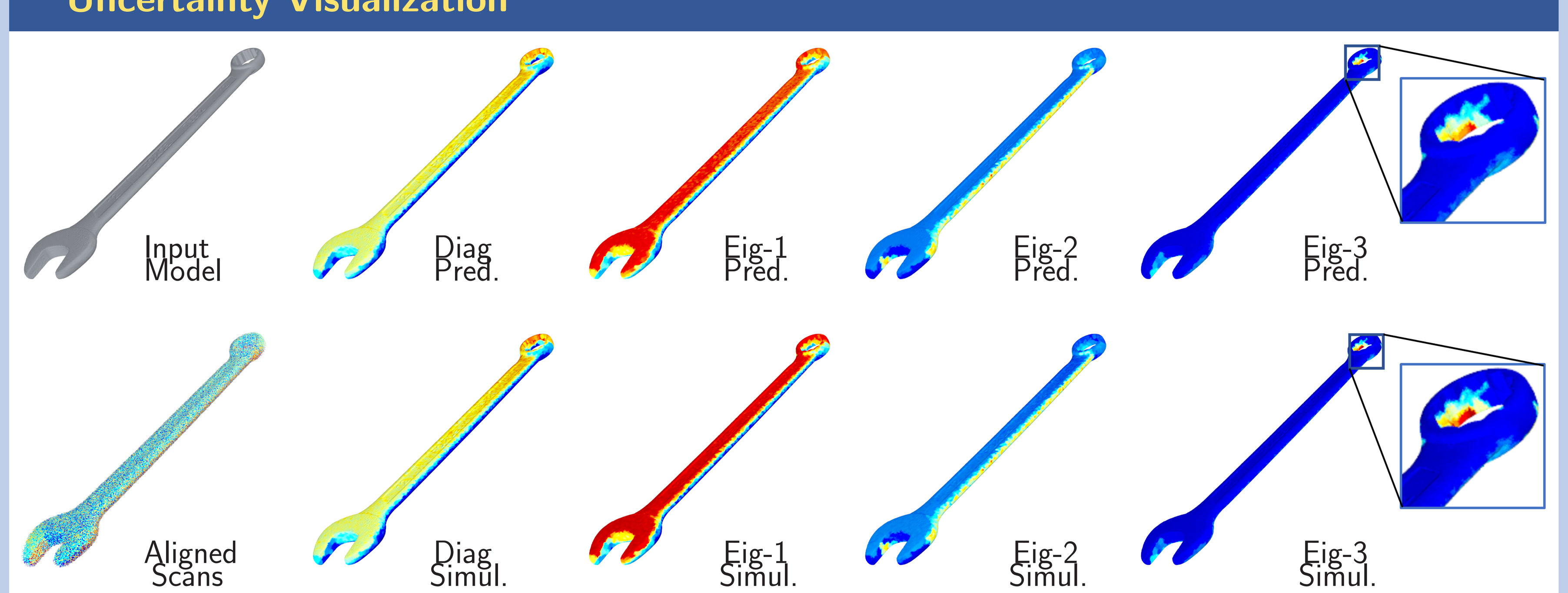

Figure: Visualizing pose uncertainties of reconstructing a wrench model under joint pairwise registration (or JPR). We show the input model, the aligned scans, the diagonal blocks, and three leading eigenvectors. The top row shows the predicted results, and the bottom row shows the simulated results. We can see that the predicted uncertainties are consistent with the intuition that this model possesses planar faces that can glide when performing registration. Such planar face structures are revealed in the leading eigenvectors. Also, our approach also accurately predicts high-frequency signals of the simulated covariance matrix.

Multi-scan Registration Results

JPR.

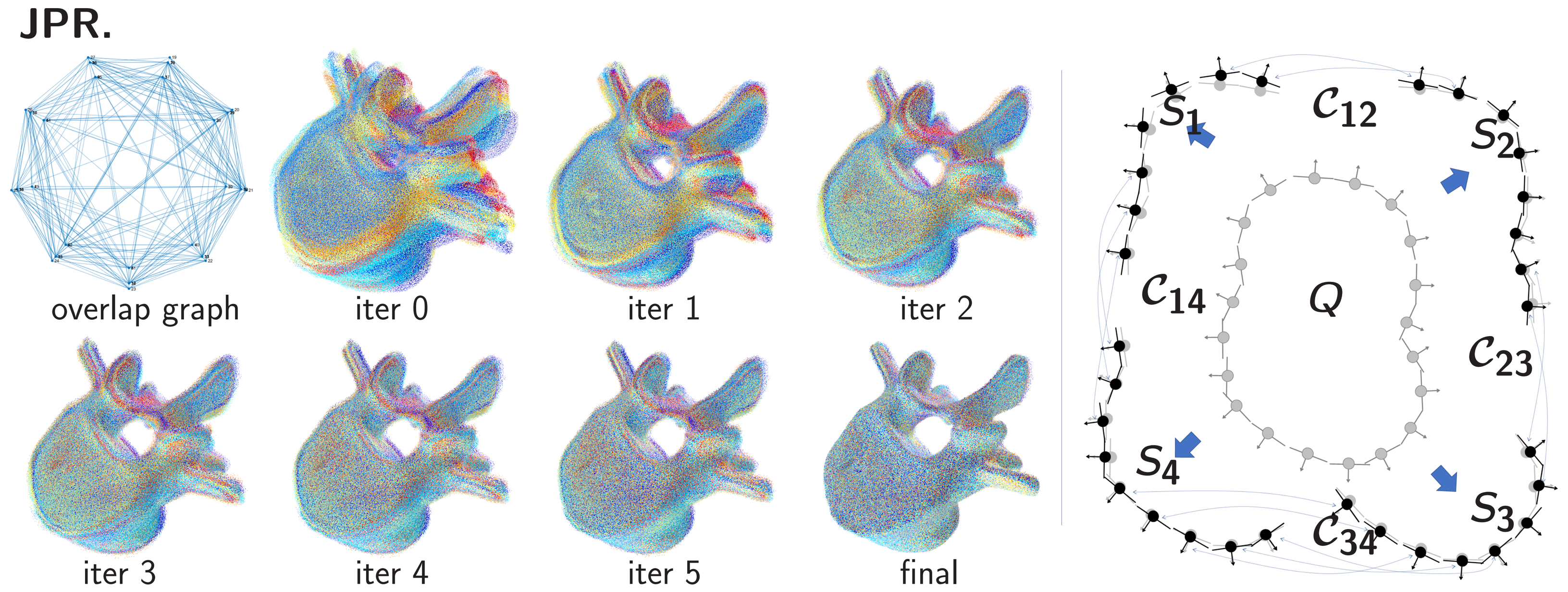

Figure: (Left) Illustration of multi-scan registration by minimizing the pairwise distances between overlapping scans. We show the overlapping graph among the input scans, intermediate results when alternating between closest-point computation and pose optimization, and the final optimized scans. (Right) Illustration of the noisy model under the setting of minimizing pairwise distances between overlapping scans. The underlying model and the ground-truth locations of the scan points are colored in gray.

SRAR.

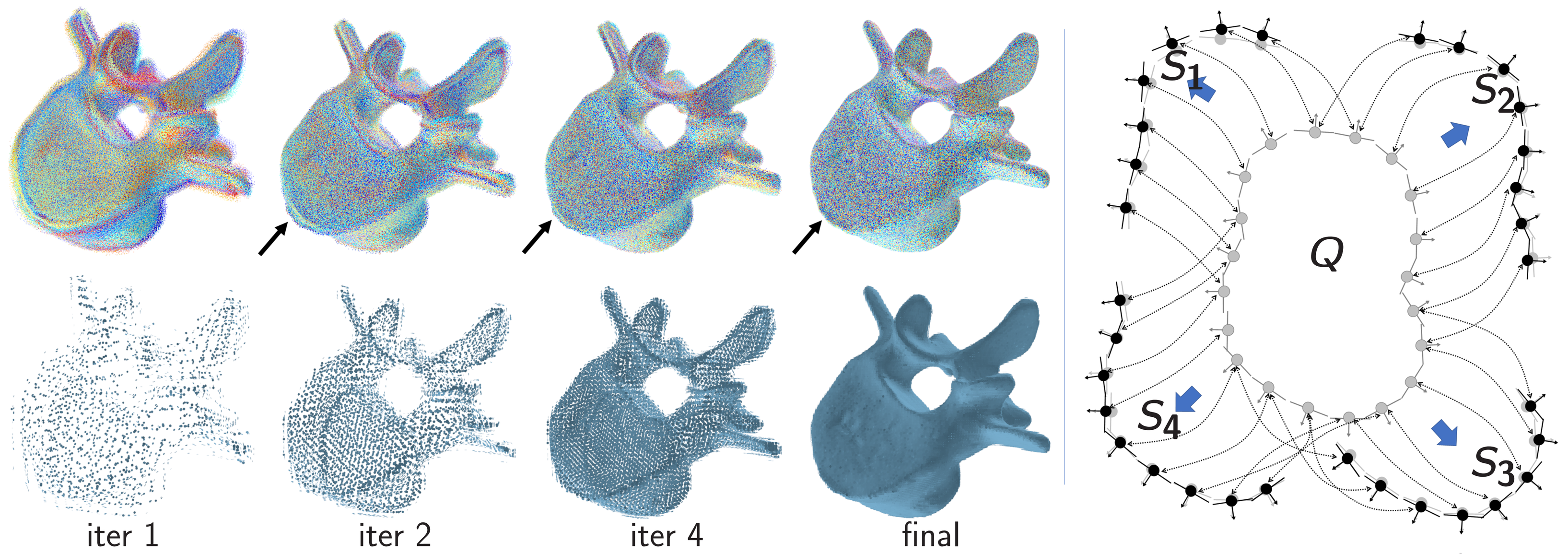

Figure: The second approach studied in this paper applies simultaneous registration and reconstruction (or SRAR) to jointly align a set of scans and reconstruct a 3D model from the aligned scans. (Left) The procedure of SRAR for reconstructing the same model in Figure 3. Each iteration optimizes the scan poses (top) and a collection of surfels (bottom) by minimizing the distances from the scan points to the surfels. These surfels are initialized by performing principal component analysis (or PCA) among scan points that fall into cells of a grid. Note that while the procedure involves registration and reconstruction at multiple levels, uncertainty quantification is performed at the finest level. (Right) An illustration of the uncertainty quantification setup for SRAR.

View Planning Results

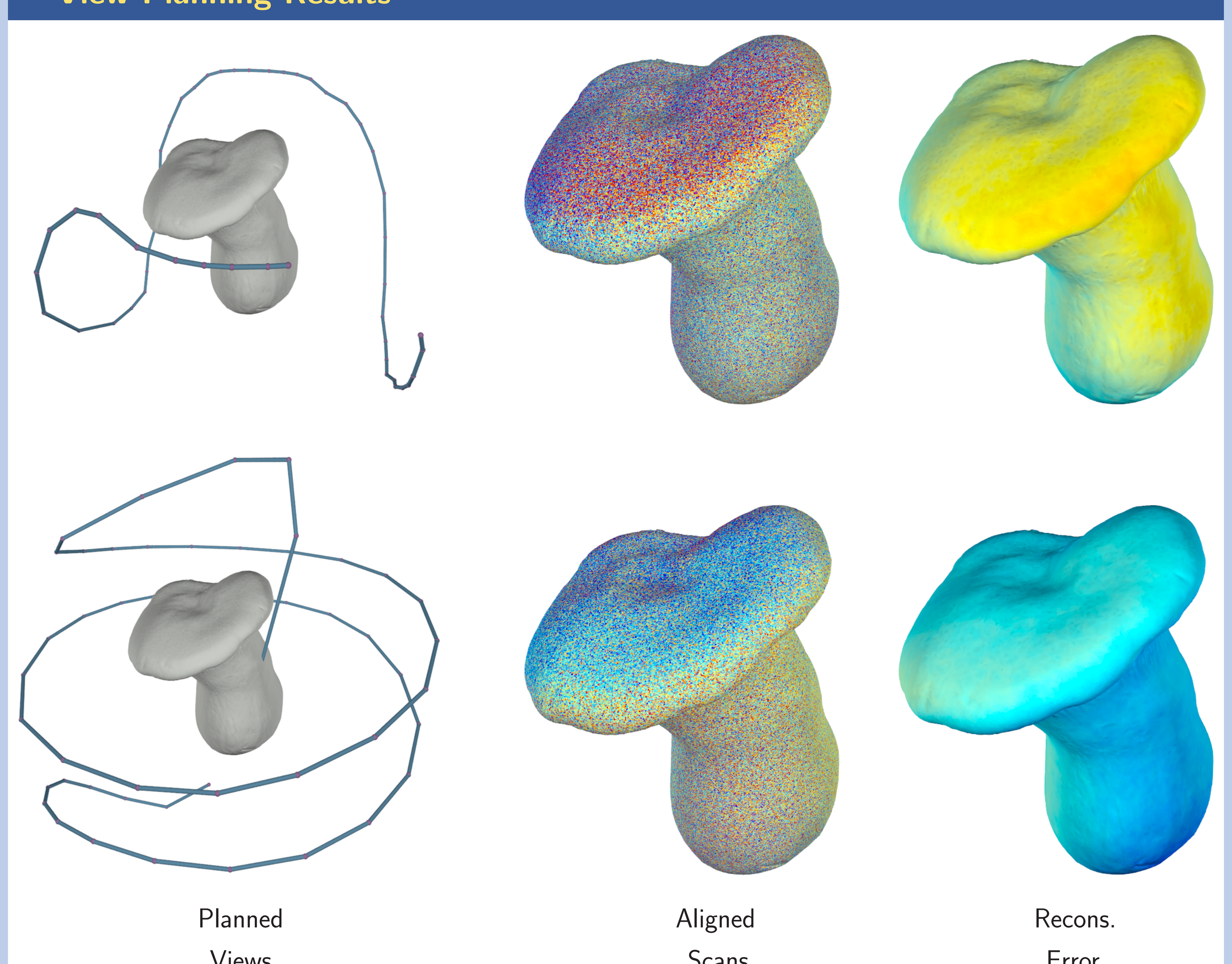

Figure: For each block, we show the planned trajectory, the resulting aligned scans, and the color-coded reconstruction error. (Top) The result obtained from setting $\lambda = 100$ in (10). (Bottom) The result obtained from setting $\lambda = 1$ in (10).