The proof of the Correctness of Convex Hull Algorithm based on M2M model

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Table 1: Terminology Explanation

Representative-Point	An arbitrary point in the part of original point set which is designated in the preprocessing.
Center-Line of two parts	The line connects the centers of two given parts.
Representative-Line	The line connects the deputies of two given parts.
Vertices of hull	The points that are used to compose the final convex hull.
Center-Hull	The convex hull of the center points of parts in a level.
Representative-Hull	The polygon that is composed by the Representative-Point belonging to Center-Hull in a level.

In order to prove the correctness of M2MCH, we first introduce four lemmas.

Lemma 1:

In the current level, all the parts whose centers are outside of the Center Hull contains no point (hull points as well) in them.

If a part contains a point but its center is outside of the convex hull of the centers in the current level, it will contradict with the definition of convex hull that requires all the points should inside the convex hull. For example, if the convex hull is given as figure XX, we can sure that there is no point in part A and Part E for their centers are outside of the convex hull.

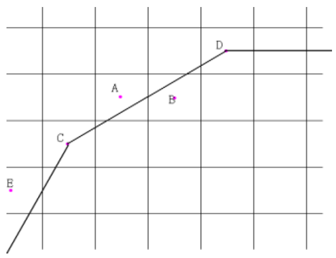

Figure 1

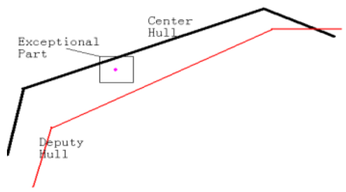

Figure 2

Lemma2:

The area inside of Representative Hull contains no hull points.

Proof:

Suppose there is a hull point lays inside of the Representative Hull, there are cases:

First case is all the hull points lay inside of the polygon of deputies. it is evident to contradict with the definition of convex hull which require all points including Representative points here shout be inside convex hull.

Second case is that some hull points lay outside of the Representative Hull. That is, there is a intersection between the convex hull and the polygon of deputies. In other words, the convex hull can not fully cover all the deputies which also contradict with its definition.

Lemma 3:

All parts have at least one intersection with the Representative Hull, if their centers are inside of Center Hull but outside of the Representative Hull. That is, there is no exceptional part as showing in figure XX.

Proof:

As shown in figure XX, D1,D2,D3,D4 present all the possible Representative Line position refer to the Center Line. Firstly, D4 is outside the Center Hull, so that the parts in the area between D4 and CenterLine needn't to be considers for they can't satisfy the precondition of lemma 3.

Then, we consider the line D1 that has a intersection with the Center Line, and the line D2 which has no intersection with the Center Line but not parallel to it, and the line D3 which is parallel to the Center Line. With the same reason as D4, the area above the Center Line and under the line D1 needn't to be considered. It is evident that if there is a exceptional part having no intersection with D1 or D2, the exceptional part also occur in the case of the line D3 which has more margin than D1 and D2. To go further, we can find the Max Margin Line which also parallel to the Center Line between two corner points(points b,d in the figure XX) . A exceptional part existing in the case of Max Margin Line is the necessary condition to that there is a exceptional part in other possible cases. Hence, if we can prove that no exceptional part exist in the case of Max Margin Line, it will be true that no exceptional part exist in other cases, that is, Lemma 3 is proved.

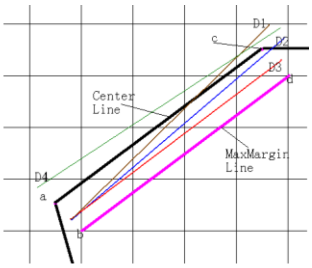

Figure 3

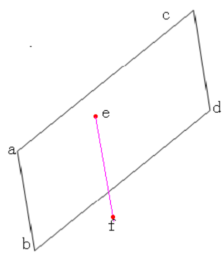

Figure 4

In the figure XX, two center points(a, c) and two corner points(b, d) form a parallelogram. And we know ab//cd, in fact, the lines between all parts' centers to their corresponding corner points are parallel to line ab and line cd, which the length of all these line is the same ($\sqrt{2}/2$ L, L is the edge length of the square part). Hence, the margin in the direction ab//cd in the parallelogram. Suppose point e is the center of the exceptional part and point f is its corresponding corner point, as shown in figure XX. We know ef//ab//cd and $|ef|=|ab|=|cd|=\sqrt{2}/2$ L. It is obvious that the possible positions of the point f is below or on the line bd which indicate the part whose center is e has at least one intersection with Representative Line bd. That is no exceptional part existing in the case of Max Margin Line Case, and Lemma 3 is proved.

Lemma 4:

All the parts which contain hull point, has at least one intersection with the Representative Hull.

Proof:

According to Lemma 1 and Lemma 2, all the hull point exist in the parts that satisfy two conditions. One condition is that those parts' centers should be inside of the Center Hull. The other is that those part should some area outside of the Representative Hull. According to Lemma 3, the parts whose centers are inside of Center Hull but outside of the Representative Hull and which also satisfy above two conditions, have been proved to have at least one intersection with the Representative Hull. The rest parts which satisfy the two conditions above but not conform with the precondition of Lemma 3, is those whose centers is inside both Center Hull and Representative Hull and it has some area outside of the Representative Hull. This kind of parts also have at least one intersection with the Representative Hull, because the center points of them inside means that some area of those parts is inside of the Representative Hull. Meanwhile, some other area of them is outside, and it is evident that there is at least intersection between those parts and the Representative Hull. Therefore, the Lemma 4 is proved.

Proof of the correctness of M2MCH:

The proof uses the following **loop invariant**:

At the end of each iteration of line5-8 in table XX, ChildSet contains all the hull points.

Initialization: The querying process begins from the top most level, which includes all the points on the original points set. Hence, at the initialization, all hull points will be included in the first level.

Maintenance: According to Lemma 4, our algorithm (Line5-6 in table XX) adds all the parts which have intersections with the Representative Hull, in other word, including all the hull points, to the ChildSet. Hence, It guarantees that the input parts set of next iteration containing all the hull points.

Termination: At the bottom level, the correct convex hull (all the hull points) is within in the final input set according to loop invariant in maintenance. The Inner algorithm can generate the correct convex hull if and only if the input set contains the desiring points. This completes the proof.