HMMT Friday Night Event
Physics is *Phun*!

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1. **Mesh of wire.** Consider a large square mesh of wire where individual squares have unit side length. Wire in the horizontal direction has resistance r per unit length, and wire in the vertical direction has resistance R per unit length. A diagonal strip of width d and length L ($L \gg d > 1$) is cut out of the mesh at an angle θ from the horizontal (fig. 1), and the two long cut edges are soldered to wire of zero resistance (the two thick lines in the figure). Find the overall resistance between the two long edges of the strip,

   (a) when $r = \infty$

   (b) in the general case.
   Hint: Your answer from part (a) may be helpful here.

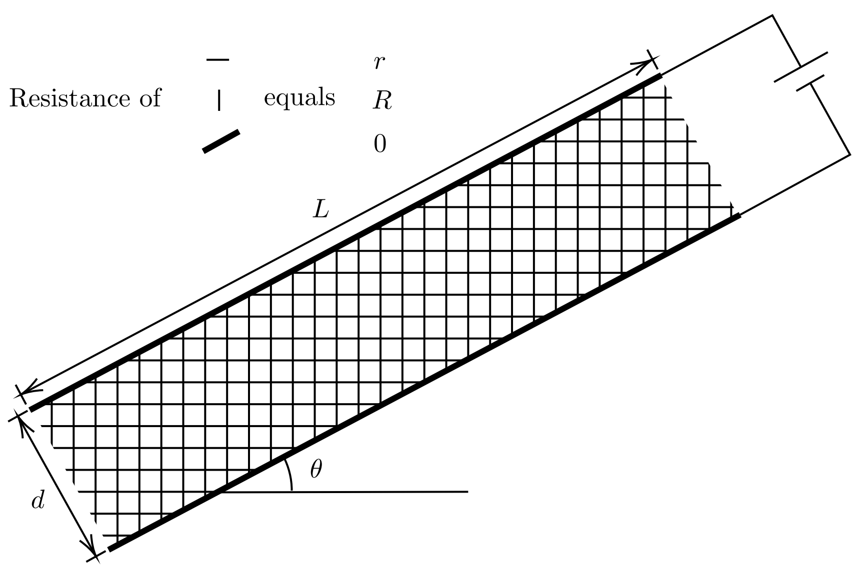


Figure 1: A diagonal strip of a grid of wire

2. **Radio antennas.** Consider a rod-shaped radio antenna with length L. Estimate $\mathrm{Re}(Z)$, the real part of the antenna's complex impedance, when it's emitting radio waves of frequency f. Write a formula in terms of L, f and physical constants, ignoring numerical constant factors.

   Recall that complex impedance Z is like resistance but for AC circuits, and in particular the analog of $P = I^2R$ for AC circuits is $P = I^2\,\mathrm{Re}(Z)$.

   Hint: The Larmor formula says that a charge q with acceleration a emits electromagnetic radiation with power $P \propto q^2a^2/\varepsilon_0 c^3$.

3. **Fried ice cream.**

   (a) What is the capacitance of a capacitor consisting of two concentric spherical metal shells of radii $a < b$? In other words, if the charge on on the inner metal shell is Q when a voltage of V is applied between the inner and outer shells, what is Q/V? Write your answer as a formula in terms of a, b, ε_0.

   (b) Fried ice cream balls are a delicious dessert consisting of an ice cream ball coated in a layer of fried crust. Model the (uncooked) dessert as an ice cream ball of radius $r = 1.5\,\text{cm}$ with latent heat of fusion $L = 3.4 \times 10^5\,\text{J/kg}$, coated in a layer of batter of uniform thickness $d = 0.5\,\text{cm}$ and thermal conductivity $\kappa = 0.6\,\text{W/m K}$. To fry it, we put it in oil of temperature $T = 200°\text{C}$. Find the rate at which the ice cream melts in units of grams per second (g/s), shortly after it starts melting in the fryer. You may make the approximation that the temperature profile of the batter/crust coating is in steady state.

   (If you don't have a calculator, write your answer as a formula in terms of $r, d, L, \kappa, \Delta T$ where $\Delta T = 200\,\text{K}$ is the temperature difference between the oil and the ice cream when it starts melting.)

   Hint: Your answer from part (a) may be useful here.

4. **Chlorophyll.** Chlorophyll is the main chemical responsible for photosynthesis in plants. To estimate the wavelength of light that chlorophyll captures, consider the simplified model where the molecule's ring is modeled as 18 electrons that are constrained to move on a circular ring of radius $r = 0.4\,\text{nm}$ (fig. 2). For simplicity, assume that the electrons do not interact, but they still must satisfy Pauli's exclusion principle.

   (a) Under the wave-particle duality, an electron with momentum p has de Broglie wavelength $\lambda = h/p$ (zero momentum corresponds to an infinite wavelength). The fact that the electron's matter wave must go through an integer number of periods around the ring quantizes the energy levels. Draw the energy level diagram and fill in the 18 electrons to create a simplified version of the molecular orbital diagram for chlorophyll. Also draw the lowest two empty energy levels.

   (b) What is the wavelength of the photon with the minimum energy required to excite chlorophyll in this simplified model? Write your answer in nanometers (nm). The mass of an electron is $m = 9.1 \times 10^{-31}\,\text{kg}$, the speed of light is $c = 3.0 \times 10^8\,\text{m/s}$, and Planck's constant is $h = 6.6 \times 10^{-34}\,\text{J s}$.
   (If you don't have a calculator, write your answer as a formula in terms of r, m, c, h.)

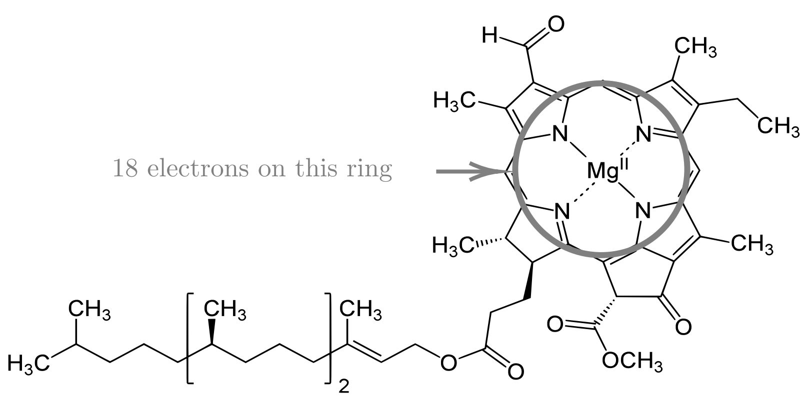


Figure 2: Chlorophyll modeled as 18 electrons moving around a ring (gray).

5. **Some geometry.**

(a) As shown in fig. 3a, $\angle AOB = 60^\circ$ and $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are tangent to circle ω, which has radius 1. Find the shortest possible length of a line segment XY (X on OA and Y on OB) that is tangent to ω.

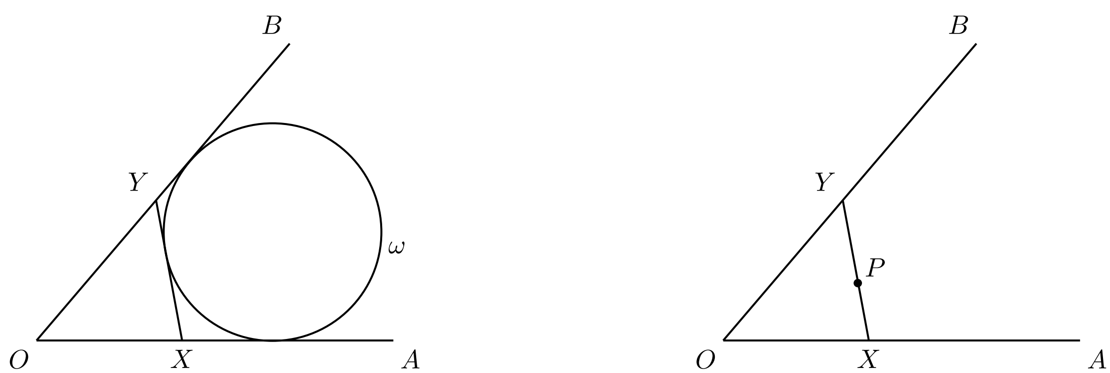
(b) As shown in fig. 3b, $\angle AOB$ and a point P inside it are fixed, whereas the line segment $\overline{XY}$ through P (X on $\overrightarrow{OA}$ and Y on $\overrightarrow{OB}$) is free to rotate around P. When the length of $\overline{XY}$ is minimized, $\overline{OX} = 15, \overline{OY} = 13, \overline{XY} = 14$. Find $\overline{PX}$.

You may find it useful to know that the altitude $\overline{OH}$ of $\triangle OXY$ has length 12.

Hint: Try to derive a somewhat simple geometric condition for minimality by analyzing the kinematics of $\overline{XY}$.

(c) Given $\triangle ABC$, find the minimum possible area of $\triangle DEF$ where D is on $\overline{BC}$, E is on $\overline{CA}$, F is on $\overline{AB}$, and $\angle D = 90^\circ - \frac{1}{2}\angle A$, $\angle E = 90^\circ - \frac{1}{2}\angle B$, $\angle F = 90^\circ - \frac{1}{2}\angle C$. Write your answer as a formula in terms of the area S, inradius r, and circumradius R of $\triangle ABC$.

Hint: Try to derive a somewhat simple geometric condition for minimality by analyzing the kinematics of $\triangle DEF$.



(a) Minimize XY subject to it being tangent to ω

(b) Minimize XY subject to it passing through P

Figure 3: Minimize length of line segment inside an angle (not drawn to scale)

Answer key:

1. (a) $\frac{d}{L}\frac{R}{\cos^2\theta}$

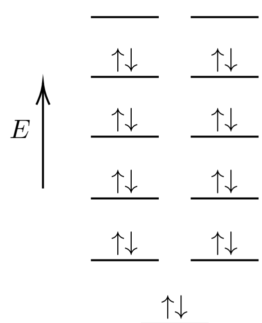
   (b) $\frac{d}{L}\frac{1}{R^{-1}\cos^2\theta + r^{-1}\sin^2\theta} = \frac{d}{L}\frac{Rr}{r\cos^2\theta + R\sin^2\theta}$

2. $\frac{L^2 f^2}{\varepsilon_0 c^3}$

3. (a) $4\pi\varepsilon_0\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = 4\pi\varepsilon_0\frac{ab}{b-a}$

   (b) $4\pi\kappa\left(\frac{1}{r} - \frac{1}{r+d}\right)^{-1}\frac{\Delta T}{L} = 4\pi\kappa\frac{r(r+d)}{d}\frac{\Delta T}{L} = 0.27\,\mathrm{g/s}$

4. (a)



   (b) $\frac{8\pi^2 cmr^2}{9h} = 580\,\mathrm{nm}$

5. (a) $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

   (b) 5

   (c) $\frac{rS}{2R}$

Tiebreaker: **Simple model of a swing.** Consider a pendulum consisting of a massless rod and a bob of mass m attached at the end. An additional mass m can be programmatically controlled to move up or down the rod by a motor. Assume the motor is so powerful that it can accelerate the mass to very high speeds almost instantly. Now, the pendulum is released from rest at an angle of $\theta_0 = 1'$ (1 arcminute, i.e., 1/60 of a degree). Your task is to control the moving mass in a way that inverts the pendulum ($\theta = 180°$) as fast as possible. What's the minimum number of full periods the pendulum must go through before it gets inverted?

Answer to tiebreaker: 6